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On infinite families of non-quasi-alternating thin knots

Tesi di Laurea Magistrale

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To His Holyness
Benedict XVI,
with mercy

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Introduction

In recent years some homology theories which are invariants of classical knots in S^3 were introduced. The most popular one is Khovanov homology (developed by Khovanov in [Kho99]), which has the crucial property that, in a suitable sense, its Euler characteristic is the Jones polynomial. After Khovanov's work, other homology theories were defined, such as odd-Khovanov homology (developed by Ozsváth, Rasmussen and Szabó in [ORS07]) and knot Floer homology (defined by Ozsváth and Szabó in [OS04a] and by Rasmussen in [Ras03]).

If K is an alternating knot, all the homology groups named above are 'simple' (in the sense that the reduced homology groups are free and supported in one single diagonal with respect to a given bigrading). The knots with 'simple' homology groups are called thin knots. Obviously, alternating knots are thin. In [MO08] Manolescu and Ozsváth proved that also quasi-alternating ($\mathcal{QA}$) knots (which are a generalization of alternating knots) are thin. A natural question is whether the converse is true.

In [Gre10] Greene proved that the thin knot 10^n_{50} is not quasi-alternating. Then, in [GW11] Greene and Watson constructed a family of thin knots K_n such that $K_0 = 11^n_{50}$ and K_n is not quasi-alternating for $n \gg 0$. Moreover, all the knots K_n have the same homological invariants mentioned above. The aim of this work is to find other infinite families of non-quasi-alternating thin knots with identical homological invariants, using the same techniques as in [GW11]. The families that we find are not defined starting from an already known non-quasi-alternating thin knot, so they provide a proof of the existence of non-quasi-alternating thin knots alternative to Greene's counterexample (cf. [Gre10]).

The techniques used to obtain this result require a lot of topological tools, introduced in Chapter 1. After recalling the definitions and the first properties of concepts such as knots and links, handle decompositions, Dehn surgery, other concepts are introduced. First, $\mathrm{Spin}^{\mathbb{C}}$ structures on a manifold are defined, both as isomorphism classes of $\mathrm{Spin}^{\mathbb{C}}(n)$-principal bundles as well as elements of the Čech cohomology group $\check{\mathrm{H}}^1(\,\cdot\,; C^\infty \mathrm{Spin}^{\mathbb{C}}(n))$. The next section deals with the concept of branched cover, with special emphasis on the branched double cover of S^3 along a link. It will be proved that, if three links are in a special relation (which is that two of them are the 'local resolutions' of the third one), then their branched double covers constitute a triad (which means that they can be obtained from each other by performing certain Dehn surgeries). The last section of Chapter 1 then gives an overview on the homology theories mentioned above and on a homology theory for 3-manifolds, called Heegaard Floer homology.

In Chapter 2 the definition of quasi-alternating link is given, and an obstruction to $\mathcal{QA}$-ness is proved: a lower bound on the correction term of the branched double cover. The correction term, in the cases we are interested in, is closely related to another important invariant of 3-manifold, which is the Turaev torsion.

The Turaev torsion is defined for an n-dimensional manifold endowed with an additional structure, called Euler structure, introduced in the first section of Chapter 3. In the case of 3-manifolds there is a canonical identification between Euler structures and $\mathrm{Spin}^{\mathbb{C}}$ structures, as proved in the second section of Chapter 3. In the last part of Chapter 3 the Turaev torsion is defined and a way to compute it is presented in the case of 3-manifolds starting from a cellular decomposition.

In Chapter 4 several families of knots are introduced. It is proved that knots belonging to the same family have the same homological invariants. Thus, if one knot is thin, so are all the knots belonging to the same family. However, the Turaev torsion of the branched double covers of the knots in a given family is not bounded from below. This implies that also the correction term is unbounded, so, by the obstruction proved in Chapter 2, infinitely many knots belonging to the family must be non-quasi-alternating. Finally, since this reasoning holds if there exists a thin knot in the family, the last part of Chapter 4 is devoted to finding such knots.

Chapter 1

Topological tools

1.1 First definitions

In this introductory section the basic definitions of Knot Theory are recalled.

Definition 1.1. Let W_1 and W_2 be two manifolds. Two C^∞ injective maps $f, g : W_1 \hookrightarrow W_2$ are called **isotopic** if for some $\varepsilon > 0$ there exists a C^∞ application $H : W_1 \times (-\varepsilon, 1+\varepsilon) \to W_2$ such that $H(\cdot, t)$ is an embedding $\forall\, t \in (-\varepsilon, 1+\varepsilon)$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ $\forall\, x \in W_1$.

Remark. The relation of isotopy is an equivalence relation.

Definition 1.2. Let Y be an oriented 3-manifold. An **oriented knot in** Y is an embedding $S^1 \hookrightarrow Y$, considered up to isotopy.

Definition 1.3. Let Y be an oriented 3-manifold. An n**-component oriented link in** Y is an embedding $\coprod_{j=1}^{n} S^1 \hookrightarrow Y$ (where $\coprod S^1$ and Y are thought of as oriented manifolds) considered up to isotopy.

Each restriction of a link $\coprod S^1 \hookrightarrow Y$ to a connected component of $\coprod S^1$ is called **component** of the link. Note that each component of a link is a knot.

Most of the times the manifold Y will be S^3. In these cases a knot (or link) in S^3 will also be called a standard knot (or standard link). Where the manifold Y is not specified, it is assumed to be S^3.

Definition 1.4. Let L be a link in S^3. $N(L)$ will denote a regular tubular neighbourhood of L. $C(L)$ will denote $S^3 \setminus N(L)$.

Definition 1.5. A **Seifert surface** for an oriented knot K is a compact connected oriented surface $S \subseteq Y$ whose boundary is K.

Definition 1.6. A **Seifert surface** for an oriented link L is a compact oriented surface $S \subseteq Y$ whose boundary is L.

Theorem 1.7 (Seifert algorithm)**.** *An oriented knot (resp. an oriented link) in S^3 always admits a Seifert surface.*

Definition 1.8. The **reverse** of a knot $K : S^1 \to Y$ is $\overline{K} = K \circ a$, where $a \in \mathrm{Diff}^-(S^1)$ (the definition does not depend on the choice of a).

Definition 1.9. The **reflection** or **obverse** of a link L is $L^r = b \circ L$, where $b \in \mathrm{Diff}^-(S^3)$ (the definition does not depend on the choice of b).

Remark. Sometimes we will not distinguish between a knot and its reverse (or, in the case of a link, between a link and the link we get reversing one of its components). In this case we will speak of unoriented knots (or links).

For a full introduction to Knot Theory, and for other definitions, such as Seifert matrix, Alexander polynomial, determinant of a link, the reader may refer to [Lic97].

1.2 Some results in low-dimensional topology

A large part of this work will require some results in low-dimensional topology (3-dimensional and 4-dimensional topology). This is the reason why this section is devoted to stating or recalling these results.

Detailed references on the topics of this section may be found in [Sco05] and [GS99].

1.2.1 Handles

Let (W, V_0, V_1) be a compact n-dimensional cobordism, i.e. an oriented n-dimensional manifold W whose oriented boundary is $V_1 \sqcup \overline{V_0}$ ($\overline{V_0}$ represents V_0 with the opposite orientation). A popular topological way to describe the cobordism is as the n-dimensional manifold $V_0 \times [0,1]$ to which a finite number of handles is attached. The definition of handle (a thickened version of a cell) can be found in [Sco05, Ch. 1, Sect. 1.2]. A handle decomposition of a cobordism always exists (cf. [Hir76, Ch. 6] for a proof in terms of Morse functions).

When dealing with handles a standard terminology is used: words like index of a handle, core, cocore, tube (or attaching tube), cotube (or belt tube), sphere (or attaching sphere), cosphere (or belt sphere) refer to precise features or subsets of a handle. This terminology is fully explained for instance in [Sco05, Ch. 1, Sect. 1.2] or (in a piecewise-linear version) in [RS72, Ch. 6]. The manifold obtained attaching a λ-handle (i.e. a handle whose index is λ) to a manifold X will be denoted by $X \cup h^\lambda$.

An n-dimesional compact manifold M can be naturally regarded as a cobordism $(M, \emptyset, \partial M)$, so the theory of handle decomposition applies also to compact manifolds with boundary. A compact manifold together with a handle decomposition is called a **handlebody**.

Recall that handles can be reordered so that each μ-handle is attached after all the λ-handles if $\lambda < \mu$, and that handles with same index are attached 'at the same time' (cf. [Sco05, Ch. 1, Sect. 1.2] or [RS72, Ch. 6]).

Let (W, V_0, V_1) be an n-dimensional cobordism with a certain handle decomposition

$$W = (V_0 \times [0,1]) \cup h_1^0 \cup \ldots \cup h_{k_0}^0 \cup \ldots \cup h_1^n \cup \ldots \cup h_{k_n}^n \cup (V_1 \times [0,1]). \quad (1.1)$$

A dual decomposition can be obtained: starting from the manifold $V_1 \times [0,1]$, all the handles of the original decomposition are attached in the opposite order; for each handle, sphere and cosphere, core and cocore, tube and cotube are swapped, the handle's index becomes $n - \lambda$ (if λ is the original index). The handle dual to h^λ will be indicated with $\overline{h^\lambda}$ (remember that it is an $(n-\lambda)$-handle). Thus, the decomposition dual to (1.1) is

$$\overline{W} = (V_1 \times [0,1]) \cup \overline{h_1^n} \cup \ldots \cup \overline{h_{k_n}^n} \cup \ldots \cup \overline{h_1^0} \cup \ldots \cup \overline{h_{k_0}^0} \cup (V_0 \times [0,1]).$$

Handle-style homology

Let (W, V_0, V_1) be an n-dimensional cobordism. It is possible to retrieve the homology of the couple (W, V_0) from a handle decomposition:

$$W = (V_0 \times [0,1]) \cup h_1^0 \cup \ldots \cup h_{k_0}^0 \cup \ldots \cup h_1^n \cup \ldots \cup h_{k_n}^n \cup (V_1 \times [0,1]).$$

Let $W^{(\lambda)}$ be the manifold obtained attaching to $V_0 \times [0,1]$ all the handles with index $\leq \lambda$:

$$W^{(\lambda)} = (V_0 \times [0,1]) \cup h_1^0 \cup \ldots \cup h_{k_0}^0 \cup \ldots \cup h_1^\lambda \cup \ldots \cup h_{k_\lambda}^\lambda.$$

The algebraic complex associated to the handle decomposition of (W, V_0, V_1) is

$$\widetilde{\mathrm{C}}_\lambda(W, V_0; R) = \mathrm{H}_\lambda(W^{(\lambda)}, W^{(\lambda-1)}; R),$$

which is naturally isomorphic to the free R-module on the λ-handles.

The boundary map $\partial_\lambda : \widetilde{\mathrm{C}}_\lambda(W, V_0; R) \to \widetilde{\mathrm{C}}_{\lambda-1}(W, V_0; R)$ is defined on the basis of $\widetilde{\mathrm{C}}_\lambda(W, V_0; R)$ given by the λ-handles as

$$\partial_\lambda(h_i^\lambda) = \sum_{j=1}^{k_{\lambda-1}} \langle h_i^\lambda | h_j^{\lambda-1} \rangle h_j^{\lambda-1},$$

where the number $\langle h_i^\lambda | h_j^{\lambda-1} \rangle$ is the intersection number between the attaching sphere of h_i^λ and the belt sphere of $h_j^{\lambda-1}$ on $\partial W^{(\lambda-1)}$ (intuitively it is the algebraic number of times that the attaching sphere of h_i^λ winds on $h_j^{\lambda-1}$).

The handle-style homology (the homology of the complex $\widetilde{\mathrm{C}}_\lambda(W, V_0; R)$) is isomorphic to $\mathrm{H}_*(W, V_0; R)$. A proof of this fact can be given adapting the proof of the equivalence of cellular and singular homology (cf. [Hat02, Ch. 2, Sect. 2.2, Cellular Homology]).

λ-handlebodies

Definition 1.10. A **λ-handlebody** (with $\lambda > 0$) is an n-manifold obtained attaching a finite number of handles whose indices are $\leq \lambda$ to a canonical ball B^n.

Remark. A λ-handlebody (if $\lambda > 0$) is always compact.

Using the handle-style homology, it can be easily checked that the homology of a λ-handlebody W obtained by attaching k λ-handles to a ball is given by:

$$\begin{cases} \mathrm{H}_0(W;R) = R \\ \mathrm{H}_\lambda(W;R) = R^k \\ \mathrm{H}_j(W;R) = 0 \qquad \text{if } j \neq 0, \lambda \end{cases}$$

A basis of the homology in dimension λ is given by the cores of the λ-handles, endowed with some orientation, to which an oriented surface $F^\lambda \subseteq B^n$ is attached (F^λ exists because the attaching sphere is 0 in $\mathrm{H}_*(B^n)$). Such a basis will be called a **standard basis** for $\mathrm{H}_\lambda(W;R)$.

The dimension of the λ-handlebodies we will deal with will always be 3 or 4. The 2-handlebodies will usually be 4-manifolds, whereas the 1-handlebodies will usually be 3-manifolds. However, the dimension of a λ-handlebody should be clear from the context.

1.2.2 A 3-manifold as the boundary of a 4-manifold

If a 3-manifold Y is the boundary of a 4-manifold X, the long exact sequence in homology of the pair (X, Y) provides a relation between the homology of Y and the homology of X. Sometimes it is therefore useful to represent a 3-manifold as the boundary of a 4-manifold. The following theorem states that not only every orientable 3-manifold is a boundary, but (provided that it is connected) it is also the boundary of a 2-handlebody with only one 0-handles and no 1-handles.

Theorem 1.11. *Let Y be a connected closed orientable 3-manifold. Then there exists a 2-handlebody with only one 0-handle and no 1-handles X such that $Y = \partial X$.*

Proof. See ([Lic97], Theorem 12.14). □

An important tool for the study of 4-manifolds is the intersection form. Before defining it, some properties of Poincaré duality are recalled. For a complete discussion on Poincaré duality, the reader may see [Hat02, Ch. 3, Sect. 3.3].

Theorem 1.12 (Poincaré-Lefschetz duality, [Hat02, Theorem 3.42])**.** *If W is a compact orientable n-manifold, and $[W, \partial W]$ is the fundamental class in $\mathrm{H}_n(W, \partial W; \mathbb{Z})$, then the maps*

$$\begin{array}{ccc} \mathrm{H}^k(W, \partial W; \mathbb{Z}) & \xrightarrow{\mathrm{PD}} & \mathrm{H}_{n-k}(W; \mathbb{Z}) \\ \varphi & \longmapsto & [W, \partial W] \frown \varphi \end{array} \qquad \begin{array}{ccc} \mathrm{H}^k(W; \mathbb{Z}) & \xrightarrow{\mathrm{PD}} & \mathrm{H}_{n-k}(W, \partial W; \mathbb{Z}) \\ \varphi & \longmapsto & [W, \partial W] \frown \varphi \end{array}$$

give isomorphisms for all k.

Proposition 1.13. *Let W be a compact orientable n-manifold. Then the following diagram (where the maps j_* and j^* are induced by the map of pairs $\mathrm{id}_W : (W, \emptyset) \to (W, \partial W)$) commutes:*

$$\begin{tikzcd} \mathrm{H}^k(W, \partial W; \mathbb{Z}) \arrow[r, "j^*"] \arrow[d, "\mathrm{PD}", "\wr"'] & \mathrm{H}^k(W; \mathbb{Z}) \arrow[d, "\mathrm{PD}", "\wr"'] \\ \mathrm{H}_{n-k}(W; \mathbb{Z}) \arrow[r, "j_*"] & \mathrm{H}_{n-k}(W, \partial W; \mathbb{Z}) \end{tikzcd}$$

Definition 1.14. A **lattice** is a couple (Z, f), where Z is a free $\mathbb{Z}$-module and f is a symmetric bilinear form on Z with coefficient in $\mathbb{Z}$.

A **homomorphism** of lattices is a map of $\mathbb{Z}$-modules

$$\varphi : (Z_1, f_1) \to (Z_2, f_2)$$

such that $\forall\, x, y \in Z_1,\ f_1(x, y) = f_2(\varphi(x), \varphi(y))$.

A **isomorphism** of lattices is a bijective homomorphism.

Definition 1.15. Let X be a compact oriented 4-manifold with boundary. The **intersection form** is the symmetric bilinear form on $\mathrm{H}_2(X; \mathbb{Z})$ defined by

$$Q_X(\alpha, \beta) = \langle \mathrm{PD}^{-1}(\alpha) \smile \mathrm{PD}^{-1}(\beta), [X, \partial X] \rangle.$$

X is **positive** (resp. **negative**) **definite** if its intersection form Q_X is so.

Lemma 1.16. *For every compact orientable 4-manifold (with boundary) X*

$$Q_X(\alpha, \beta) = \langle j^* \circ \mathrm{PD}^{-1}(\alpha), \beta \rangle.$$

Proof. The relation between cap product and cup product (cf. [Hat02, Ch. 3, Sect. 3.3, Connection with Cup Product]) implies that

$$\begin{aligned} Q_X(\alpha, \beta) &= \langle \mathrm{PD}^{-1}(\alpha) \smile \mathrm{PD}^{-1}(\beta), [X, \partial X] \rangle \\ &= \langle j^* \circ \mathrm{PD}^{-1}(\alpha), [X, \partial X] \frown \mathrm{PD}^{-1}(\beta) \rangle. \end{aligned}$$

As $[X, \partial X] \frown \mathrm{PD}^{-1}(\beta)$ is the Poincaré dual of $\mathrm{PD}^{-1}(\beta)$ (which is β itself), the Lemma is proved. $\square$

Lemma 1.17. *If A and B are two surfaces properly embedded in X, then*

$$Q_X([A],[B]) = \text{algebraic intersection number of } A \text{ and } B.$$

Proof. In [Sco05, Ch. 3, Sect. 3.2] there is the proof if X is a closed manifold. However, the proof can be easily adapted for the general case noting that, if the surfaces are transversely properly embedded, there is no intersection point in ∂X. □

If Y is the boundary of a 2-handlebody with no 1-handles X, there is a relation between $\mathrm{H}_1(Y;\mathbb{Z})$ and the intersection form of X.

Lemma 1.18. *Let X be a 2-handlebody with only one 0-handle and no 1-handles, with $Y = \partial X$. Then the matrix of the intersection form Q_X is a presentation matrix for $\mathrm{H}_1(Y;\mathbb{Z})$.*

Proof. Consider the long exact sequence of the pair (X,Y):

$$\mathrm{H}_2(X;\mathbb{Z}) \xrightarrow{j_*} \mathrm{H}_2(X,Y;\mathbb{Z}) \longrightarrow \mathrm{H}_1(Y;\mathbb{Z}) \longrightarrow \mathrm{H}_1(X;\mathbb{Z}).$$

Since $\mathrm{H}_1(X;\mathbb{Z}) = 0$ (because X has no 1-handles), a matrix representing j_* is a presentation matrix for $\mathrm{H}_1(Y;\mathbb{Z})$.

In order to calculate a matrix representing j_*, recall that $\mathrm{H}_2(X;\mathbb{Z}) \cong \mathbb{Z}^n$, where n is the number of the 2-handles. Let $e_1,\ldots,e_n$ be the standard basis of $\mathrm{H}_2(X;\mathbb{Z})$ (i.e. the one given by the oriented cores of the 2-handles to which a Seifert surface of the attaching sphere is attached). Let $\{e_1^*,\ldots,e_n^*\}$ be the basis of $(\mathrm{H}_2(X;\mathbb{Z}))^*$ dual to $\{e_1,\ldots,e_n\}$.

The Universal Coefficient Theorem (cf. [Hat02, Theorem 3.2])

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_1(X;\mathbb{Z}),\mathbb{Z}) \longrightarrow \mathrm{H}^2(X;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_2(X;\mathbb{Z}),\mathbb{Z}) \longrightarrow 0$$

together with the fact that $\mathrm{H}_1(X;\mathbb{Z}) = 0$ proves that the evaluation morphism $\mathrm{ev} : \mathrm{H}^2(X;\mathbb{Z}) \to (\mathrm{H}_2(X;\mathbb{Z}))^*$ is an isomorphism.

Consider now the composition map

$$(\mathrm{H}_2(X;\mathbb{Z}))^* \xrightarrow{\mathrm{ev}^{-1}} \mathrm{H}^2(X;\mathbb{Z}) \xrightarrow{\mathrm{PD}} \mathrm{H}_2(X,Y;\mathbb{Z}).$$

As ev and PD are isomorphisms, so is $\mathrm{PD}\circ\mathrm{ev}^{-1}$, and the basis $\{e_1^*,\ldots,e_n^*\}$ maps through $\mathrm{PD}\circ\mathrm{ev}^{-1}$ to a basis $\{\varepsilon_1,\ldots,\varepsilon_n\}$ of $\mathrm{H}_2(X,Y;\mathbb{Z})$.

Let a_{ki} be the coordinates of $j_*(e_i)$ with respect to the basis $\varepsilon_1,\ldots,\varepsilon_n$:

$$j_*(e_i) = \sum_{k=1}^{n} a_{ki}\varepsilon_k.$$

This means that

$$a_{ki} = (\mathrm{ev}\circ\mathrm{PD}^{-1}\circ j_*(e_i))(e_k) = \langle \mathrm{PD}^{-1}\circ j_*(e_i), e_k\rangle.$$

By Proposition 1.13 and Lemma 1.16

$$a_{ki} = \langle j^* \circ \mathrm{PD}^{-1}(e_i), e_k \rangle = Q_X(e_i, e_k),$$

so the matrix representing Q_X is the matrix representing j_*, thus, it is a presentation matrix for $\mathrm{H}_1(Y;\mathbb{Z})$. □

Definition 1.19. Let Y be a 3-manifold. $|\,\mathrm{H}_1(Y;\mathbb{Z})|$ denotes the cardinality of the set $\mathrm{H}_1(Y;\mathbb{Z})$, provided that it is finite, and $|\,\mathrm{H}_1(Y;\mathbb{Z})| = 0$ otherwise.

Corollary 1.20. *Let X be a 2-handlebody with only one 0-handle and no 1-handles, with $Y = \partial X$, and let Q_X be the matrix of the intersection form. Then*

$$|\det Q_X| = |\,\mathrm{H}_1(Y;\mathbb{Z})|. \tag{1.2}$$

Proof. It directly follows from the fact that Q_X is a presentation matrix for $\mathrm{H}_1(Y;\mathbb{Z})$ (cf. Lemma 1.18). Note that Equation (1.2) holds also if $\det Q_X = 0$ (cf. Definition 1.19). □

1.2.3 Dehn filling

A very popular way to construct new 3-manifolds starting from a given one is through performing a Dehn surgery or filling. The importance of this construction is that every connected orientable closed 3-manifold can be constructed by performing a finite number of Dehn surgeries on S^3 (as we will see, this is a restatement of Theorem 1.11).

Definition 1.21. Let Y be a 3-manifold, whose boundary has a toric component Z. A Dehn filling on Y along Z is the 3-manifold obtained by attaching Y and a solid torus $S^1 \times D^2$ through an orientation-reversing diffeomorphism between Z and $\mathbb{T} = \partial(S^1 \times D^2)$.

Remark. The Dehn filling clearly preserves orientability, connectedness and compactness of a manifold.

Proposition 1.22. *Let Y be a 3-manifold, whose boundary has a toric component Z. Suppose that two homology classes λ and μ in $\mathrm{H}_1(Z;\mathbb{Z})$ such that $\#(\lambda \cap \mu) = 1$ are given, and let us call them longitude and meridian. Let $\varphi : \mathbb{T} = \partial(S^1 \times D^2) \to Z$ be the orientation-reversing attaching diffeomorphism and let m and l be a meridian and a longitude in $\mathrm{H}_1(\mathbb{T};\mathbb{Z})$. Let a and b be natural numbers such that in $\mathrm{H}_1(Z;\mathbb{Z})$*

$$\varphi_*(m) = a\mu + b\lambda.$$

Then the manifold obtained by Dehn filling on Y along Z is determined (up to diffeomorphism) by the number $a/b \in \mathbb{Q} \cup \{\pm\infty\}$. This number is called the (rational) ***framing*** *of the Dehn filling.*

Sketch of the proof. Consider the map φ_* in homology. Choose as bases of $\mathrm{H}_1(\mathbb{T};\mathbb{Z})$ and $\mathrm{H}_1(Z;\mathbb{Z})$ the ones given by meridian and longitude $\{m,l\}$ and $\{\mu,\lambda\}$. The determinant of the map φ_* with respect to these bases must be 1. Hence, if $\varphi_*(m) = a\mu + b\lambda$ and $\varphi_*(l) = s\mu + t\lambda$, the equation $at - bs = 1$ must be satisfied.

Suppose that a and b are fixed. Bézout's Theorem assures that numbers s and t in $\mathbb{Z}$ such that $at - bs = 1$ do exist. However, they are not unique. Let s' and t' be other integers such that $at' - bs' = 1$. It is easy to check (using the fact that a and b are relatively prime) that $s - s' = na$ and $t - t' = nb$ for some $n \in \mathbb{Z}$. As a result the image of the meridian $\varphi_*(m)$ defines the image of the longitude $\varphi_*(l)$ up to multiples of $\varphi_*(m)$.

Now, the diffeomorphism φ is defined (up to isotopy) by φ_*, and so the manifold $Y \cup_\varphi (S^1 \times D^2)$ depends (up to diffeomorphism) only on φ_*. Moreover, choosing φ_1 and φ_2 such that

$$\begin{aligned} (\varphi_1)_*(m) &= a\mu + b\lambda & \qquad (\varphi_2)_*(m) &= a\mu + b\lambda \\ (\varphi_1)_*(l) &= s\mu + t\lambda & \qquad (\varphi_2)_*(l) &= (s+na)\mu + (t+nb)\lambda \end{aligned}$$

the resulting manifolds $Y \cup_{\varphi_1} (S^1 \times D^2)$ and $Y \cup_{\varphi_2} (S^1 \times D^2)$ are diffeomorphic (the diffeomorphism is obtained just performing n twists on $S^1 \times D^2$ before attaching).

It is worth noting also that the choice of the longitude l in $\mathbb{T}$ (which is defined only up to multiples of m) is not relevant, as a longitude can be carried to any other just performing some twists on $S^1 \times D^2$.

To sum up, the manifold $Y \cup_\varphi (S^1 \times D^2)$ depends only on $\varphi_*(m) = a\mu + b\lambda$ (where a and b are relatively prime). To see that it actually depends only on a/b it suffices to note that changing the signs of both a and b equals to reversing both the meridian and the longitude on $\mathbb{T}$, so the resulting manifold is the same. □

Definition 1.23. A **Dehn surgery** on a 3-manifold Y consists of removing a solid torus $S^1 \times D^2$ from Y and performing a Dehn filling on $\partial(S^1 \times D^2)$.

Remark. When performing a Dehn surgery, the meridian μ of $\partial(S^1 \times D^2)$ is well defined (it is the boundary of a disk in the removing solid torus $S^1 \times D^2$), whereas the longitude is defined only up to multiples of the meridian.

Definition 1.24. A Dehn filling (or Dehn surgery) on Y is called **longitudinal** if the associated rational number a/b is an integer (i.e. if the meridian m of the attaching solid torus is sent to a longitude of the attaching torus in ∂Y.

Remark. In the case of longitudinal Dehn filling (or surgery) the choice of the framing (i.e. the rational number a/b) corresponds to the choice of a longitude of the attaching torus in ∂Y.

Remember that a given longitude of the attaching torus in ∂Y must already exist to define the framing as a rational (actually integer) number.

In the case of longitudinal Dehn filling (or surgery) the framing can directly be defined as a longitude, without the need of choosing a 'referring' longitude and a number.

Remark. A longitudinal Dehn surgery on a 3-manifold Y can be represented as a knot K in Y together with a longitude λ. A regular neighbourhood of K is the removing solid torus, and the longitude λ is the longitude where the meridian of the attaching solid torus is mapped to. λ is also called a **framing** for K.

If $Y = S^3$ and $K \subseteq S^3$ is a knot, a preferred longitude λ^S is given (i.e. the boundary of a Siefert surface), so a longitude λ can be represented as an integer $f \in \mathbb{Z}$, in such a way that

$$\lambda = \lambda^S + f\mu,$$

where μ is a meridian. If a Dehn surgery is performed on (K, λ), the number f is exactly the framing of the Dehn surgery.

A 3-dimensional operation as performing a longitudinal Dehn surgery on a manifold Y is closely connected with a 4-dimensional operation, that is attaching a 2-handle to a 4-manifold whose boundary is Y.

Lemma 1.25. *Let X be a 4-manifold whose boundary is a 3-manifold Y. Let (K, λ) be a framed knot (i.e. a knot with a framing) in Y. The boundary of the manifold $X \cup h^2$, where h^2 is a 2-handle attached in such a way that K is its attaching sphere and λ is the boundary of a disk parallel to the core, is obtained from Y by Dehn surgery along K with framing λ.*

Proof. The boundary of $X \cup h^2$ is obtained from Y removing a neighbourhood of the attaching sphere K and replacing it with a solid torus $S^1 \times D^2$ whose boundary is identified with the boundary of the neighbourhood of K. The longitude λ bounds a disk in the attaching solid torus, so it is the image of the meridian of the attaching solid torus through the attaching map.

Thus, $\partial(X \cup h^2)$ is obtained by longitudinal Dehn surgery from the manifold Y on the pair (K, λ). □

Definition 1.26. Let L be a framed link in S^3 (i.e. a link with a framing for each component). The pair (X_L, S^3_L) is obtained by **L-surgery** from $S^3 = \partial B^4$ if $S^3_L = \partial X_L$ and X_L is the 4-manifold obtained by attaching a 2-handle to B^4 for every component K of L in such a way that the attaching sphere is K itself and the attaching framing is the framing of K.

Remark. Let K be a knot in a general 3-dimensional manifold Y. An analogous definition can be introduced in this case. Specifically, a framing of K in this case is a given longitude λ (in a general manifold there is not a preferred longitude), and we say that the manifold Y_λ is obtained by **λ-surgery** from Y if it is the result of a Dehn surgery along K with framing λ (i.e. so that λ is the identified with the meridian of the attaching torus).

Proposition 1.27. *Let* $L = K_1 \cup \ldots \cup K_n$ *be a link in* $S^3 = \partial B^4$ *with* n *components, with framings* $f_1, \ldots, f_n$. *Let* (X, S^3_L) *be the pair obtained by L-surgery. Let* $\eta_1, \ldots, \eta_n$ *be 2-cycles that induce a standard basis of* $\mathrm{H}_2(X;\mathbb{Z})$ *(i.e. 2-cycles corresponding to oriented cores of the 2-handles* $h^2_1, \ldots, h^2_n$, *to each of which a Seifert surface in* S^3 *is attached).*

Then, the intersection form (with respect to this basis) is given by the $n \times n$ *matrix*

$$A = \big(\mathrm{lk}(K_i, K_j)\big)_{i,j=1,\ldots,n},$$

where we define $\mathrm{lk}(K_i, K_i) = f_i$.

Proof. For each i, let D_i be the core of the handle h_i, so that $\partial D_i = K_i$. Let S_i be a Seifert surface for K_i. Then

$$\eta_i = D_i \cup \overline{S_i},$$

where $\overline{S_i}$ denotes S_i with reversed orientation.

Moreover, let e_{K_i} be the vector tangent to the oriented knot K_i, and let e_{r_i} be the radial vector in D_i, so that $\{e_{r_i}, e_{K_i}\}$ gives the given orientation of D_i (cf. Figure 1.1).

Suppose $i \neq j$. In order to calculate $\#(\eta_i \cap \eta_j)$, push $\overline{S_i}$ inside D^4, as suggested in Figure 1.1. After this small perturbation, the intersection between η_i and η_j completely lies in S^3, and it is the same as the intersection between K_i and $\overline{S_j}$. Focus on a point p of the intersection. The sign of the intersection is the sign of the basis of the tangent space obtained juxtaposing the basis $\{e_{r_i}, e_{K_i}\}$ of D_i and the opposite of a basis $\{x_{S_j}, y_{S_j}\}$ of S_j:

$$\varepsilon(p) = \varepsilon_{B^4}(\{e_{r_i}, e_{K_i}, -x_{S_j}, y_{S_j}\}).$$

As suggested by Figure 1.1, the outer normal (exiting from B^4) in p is $-e_{r_i}$. According to the 'outer normal first' rule,

$$\varepsilon(p) = \varepsilon_{B^4}(\{-e_{r_i}, e_{K_i}, x_{S_j}, y_{S_j}\}) = \varepsilon_{S^3}(\{e_{K_i}, x_{S_j}, y_{S_j}\}),$$

which is the sign of p as intersection of K_i and S_j. Thus, the sum over all the intersection points gives exactly the linking number $\mathrm{lk}(K_i, K_j)$.

The case $i = j$ is very similar to the previous one. Just perturb one of the two copies of η_i pushing the core D_i of the handle h^2_i on the boundary and the surface $\overline{S_i}$ inside the ball B^4. Now the situation is very similar to the previous one, with the difference that now $\eta_i \cap \eta_j$ coincides with the intersection of $\overline{S_j}$ and the longitude of K_i determined by the attaching framing, i.e. the longitude $\lambda^S_i + f_i\mu_i$, where μ_i is a meridian of K_i and λ^S_i is the standard longitude in S^3, determined by a Seifert surface. In this case the intersection number is (reasoning as above)

$$\mathrm{lk}(\lambda^S_i + f_i\,\mu_i, K_i) = \mathrm{lk}(\lambda^S_i, K_i) + f_i\,\mathrm{lk}(\mu_i, K_i) = 0 + f_i \cdot 1 = f_i.$$

The proof of the proposition is then complete. $\square$

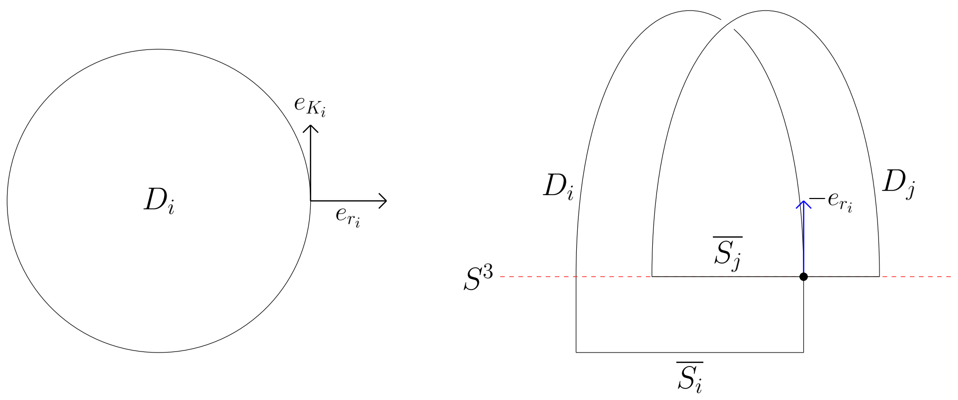


Figure 1.1: The picture on the left represents the core D_i of the handle h_i^2. e_{K_i} is the vector tangent to the oriented knot K_i, whereas e_{r_i} is the radial vector. The basis $\{e_{r_i}, e_{K_i}\}$ gives the orientation of D_i. The picture on the right conveys the idea of how the cycles η_i and η_j intersect (the cycles are actually 2-dimensional, and not 1-dimensional as the picture seems to suggest). The dashed red line represents S^3, the part below it B^4, and the part above it the attaching handles. The cycle η_i is pushed inside B^4, so that the intersection between η_i and η_j is exactly the intersection between K_i and $\overline{S_j}$.

1.3 Spin$^{\mathbb{C}}$ structures

Let W be a (not necessarily closed) manifold. In this section we will study an additional structure (a Spin$^{\mathbb{C}}$ structure) that W may be endowed with. A full introduction to Spin$^{\mathbb{C}}$ structures can be found in [Sco05, Ch. 10, Sect. 10.2].

1.3.1 Principal bundles and Čech cocycles

The present subsection deals quickly with the concept of principal bundle and with the presentation of a principal bundle through a Čech cocycle. A complete discussion on the topic can be found in [Sco05, Ch. 4, Note: Bundles, cocycles, and Čech cohomology].

Definition 1.28. Let W be a manifold and G be a Lie group. A **G-principal bundle** is a C^∞ map $\eta : \mathcal{P}_G \to W$ such that each fiber $\eta^{-1}(x)$ is endowed with a free and transitive C^∞ right action of G and $\forall\, x \in W$ there exist a neighbourhood U of x and a G-equivariant diffeomorphism $\psi : \eta^{-1}(U) \to U \times G$ (where G acts on $U \times G$ trivially on the first component and as the right multiplication on the second one), such that the following diagram commutes:

$$\begin{tikzcd} \eta^{-1}(U) \arrow[rr, "\cong", "\psi"'] \arrow[dr, "\eta"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

The set U is called a trivializing set for η, and the map ψ is called a trivialization of η on U.

Example 1.29. Let W be an oriented n-dimensional manifold, and let TW be its tangent bundle. Choose a riemannian structure on W, and let $\mathcal{P}_{\mathrm{SO}(n)}$ be the bundle of oriented orthonormal frames of TW (an oriented orthonormal frame can be thought of as an orientation-preserving isometry $\beta : \mathbb{R}^n \to T_xW$). Then, $\mathcal{P}_{\mathrm{SO}(n)}$ is an $\mathrm{SO}(n)$-principal bundle with the action given by the composition of β and an isometry f of $\mathbb{R}^n$:

$$\beta \cdot f = \beta \circ f.$$

Example 1.30. More generally speaking, let $E \to W$ be an oriented real vector bundle of rank r. The $\mathrm{SO}(r)$-bundle $\mathrm{Iso}_W(\mathbb{R}^r \times W, E)$ of the orientation-preserving isometries between of $\mathbb{R}^r \times W$ and E (i.e. the homomorphisms of real vector bundles that are orientation-preserving isometries on each fiber) is an $\mathrm{SO}(r)$-principal bundle.

Conversely, given an $\mathrm{SO}(r)$-principal bundle $\mathcal{P}_{\mathrm{SO}(r)}$, it is possible to construct an associated oriented real vector bundle as the quotient of $\mathcal{P}_{\mathrm{SO}(r)} \times \mathbb{R}^r$ by the action of $\mathrm{SO}(r)$ (which is a right action on the component $\mathcal{P}_{\mathrm{SO}(r)}$ and the obvious left action on $\mathbb{R}^r$).

The two maps

$$(E \to W) \longmapsto \mathrm{Iso}_W(\mathbb{R}^r \times W, E)$$
$$\mathcal{P}_{\mathrm{SO}(r)} \underset{\mathrm{SO}(r)}{\times} \mathbb{R}^r \longleftarrow\!\shortmid \mathcal{P}_{\mathrm{SO}(r)}$$

are one the inverse of the other and furnish a bijection between isomorphism classes of oriented real vector bundles of rank r and $\mathrm{SO}(r)$-principal bundles.

Definition 1.31. Two G-principal bundles $\eta : \mathcal{P}_G \to W$ and $\overline{\eta} : \overline{\mathcal{P}_G} \to W$ are **isomorphic** if there exists a G-equivariant diffeomorphism $\varphi : \mathcal{P}_G \to \overline{\mathcal{P}_G}$ that preserves the fibers, i.e. such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{P}_G \arrow[rr, "\cong", "\varphi"'] \arrow[dr, "\eta"'] & & \overline{\mathcal{P}_G} \arrow[dl, "\overline{\eta}"] \\ & W & \end{tikzcd}$$

Definition 1.32. Let $\eta_G : \mathcal{P}_G \to W$ be a G-principal bundle. A surjective group homomorphism $\sigma : G \to H$ naturally defines an H-principal bundle $\eta_H : \mathcal{P}_H \to W$ as the fiberwise quotient ${}^{\mathcal{P}_G}/_{\ker\sigma}$. The bundles obviously fit the following diagram:

$$\begin{tikzcd} \mathcal{P}_G \arrow[rr, "\pi"'] \arrow[dr, "\eta_G"'] & & \mathcal{P}_H \arrow[dl, "\eta_H"] \\ & W & \end{tikzcd}$$

The quotient map π is $(\sigma : G \to H)$-equivariant, i.e. $\forall\, x \in \mathcal{P}_G$ and $q \in G$

$$\pi(x \cdot q) = \pi(x) \cdot \sigma(q).$$

If $(U_\alpha)_{\alpha\in A}$ is an acyclic cover of a manifold W, a G-principal bundle $\eta : \mathcal{P}_G \to W$ can be represented in terms of transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G.$$

If $\psi_\alpha : \eta^{-1}(U_\alpha) \to U_\alpha \times G$ and $\psi_\beta : \eta^{-1}(U_\beta) \to U_\beta \times G$ are local trivializations of η as in Definition 1.28, then $g_{\alpha\beta}$ is defined as follows:

$$\psi_\alpha \circ \psi_\beta^{-1}{}_{|\psi_\beta(U_\alpha\cap U_\beta)}(x, q) = (x, g_{\alpha\beta}(x) \cdot q).$$

Just like the transition functions of a vector bundle, $\forall\, \alpha, \beta, \gamma \in A$

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1.$$

This is the reason why the following definition is introduced.

Definition 1.33. Let G be a Lie group, W a manifold, and let $(U_\alpha)_{\alpha\in A}$ be an acyclic cover of W.

A **Čech 1-cocycle** is a collection of C^∞ maps $(g_{\alpha\beta} : U_\alpha \cap U_\beta \to G)_{\alpha,\beta\in A}$ such that $\forall\, \alpha, \beta, \gamma \in A$

$$g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1. \tag{1.3}$$

Two cocycles $(g_{\alpha\beta})_{\alpha,\beta\in A}$ and $(g'_{\alpha\beta})_{\alpha,\beta\in A}$ are cohomologous if there exists a collection of C^∞ maps $(f_\alpha : U_\alpha \to G)_{\alpha\in A}$ such that $\forall\, \alpha, \beta \in A$

$$g'_{\alpha\beta} = f_\alpha\, g_{\alpha\beta}\, f_\beta^{-1}.$$

The set of equivalence classes of Čech cocycles, equipped with the operation induced by the multiplication on G, is called the **first Čech cohomology group** of W with coefficients in G, and it is denoted by $\check{\mathrm{H}}^1(W; C^\infty G)$.

The definition of $\check{\mathrm{H}}^1(W; C^\infty G)$ does not depend on the choice of the acyclic cover of W since the cohomology groups obtained from two acyclic covers are canonically and functorially isomorphic.

Now let $\mathcal{P}_G$ be a G-principal bundle over W, and let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a open covering of W trivializing for $\mathcal{P}_G$. The transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

of the principal bundle satisfy the cocycle condition (1.3) and hence they represent a class in $\check{\mathrm{H}}^1(W; C^\infty G)$. This map actually yields a bijection between $\check{\mathrm{H}}^1(W; C^\infty G)$ and the set of G-principal bundles up to isomorphism, as the following Lemma states.

Lemma 1.34. *Let W be a manifold and G be a Lie group. There exists a bijection between $\check{\mathrm{H}}^1(W; C^\infty G)$ and the isomorphism classes of G-principal bundles over W.*

The bijection simply maps a Čech cocycle $(g_{\alpha\beta})$ to the G-principal bundle whose transition functions are exactly $(g_{\alpha\beta})$.

Proof. The proof of this fact can be found in [Sco05, Ch. 4, Note: Bundles, cocycles, and Čech cohomology]. □

1.3.2 Spin$^{\mathbb{C}}$ structures

Definition 1.35. Let $r \in \mathbb{N}, r > 1$. The Lie group $\mathrm{Spin}(r)$ is the two-fold cover of the group $\mathrm{SO}(r)$. As a result there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(r) \xrightarrow{\rho} \mathrm{SO}(r) \longrightarrow 0.$$

Remark. If $r > 2$, $\mathrm{Spin}(r)$ is the universal cover of $\mathrm{SO}(r)$.

Definition 1.36. Let $r \in \mathbb{N}, r > 1$. The group $\mathrm{Spin}^{\mathbb{C}}(r)$ is defined as

$$\mathrm{Spin}^{\mathbb{C}}(r) = \frac{\mathrm{U}(1) \times \mathrm{Spin}(r)}{\{(1,1), (-1,-1)\}}.$$

Remark. There exists a short exact sequence

$$0 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^{\mathbb{C}}(r) \xrightarrow{\sigma} \mathrm{SO}(r) \longrightarrow 0,$$

where σ is defined as

$$\sigma : [\zeta, h] \longmapsto \rho(h).$$

Definition 1.37. Let $E \to W$ be an oriented real vector bundle of rank r on an n-dimensional manifold. A pair (η, H) consists of a $\mathrm{Spin}^{\mathbb{C}}(r)$-principal bundle $\eta : \mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)} \to W$ and an isomorphism H between $\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}/\mathrm{U}(1)$ and the bundle $\mathcal{P}_{\mathrm{SO}(r)}$ associated to $E \to M$ (cf. Example 1.30).

$$
\begin{tikzcd}
\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)} \arrow[rrrr, "\cong", "\Psi"'] \arrow[d] \arrow[ddrr, "\eta"'] & & & & \overline{\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}} \arrow[d] \arrow[ddll, "\overline{\eta}"] \\
\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}/\mathrm{U}(1) \arrow[rr, "\cong", "H"'] \arrow[drr] & & \mathcal{P}_{\mathrm{SO}(r)} \arrow[d] & & \overline{\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}}/\mathrm{U}(1) \arrow[ll, "\cong"', "\overline{H}"] \arrow[dll] \\
& & W & &
\end{tikzcd}
$$

Diagram 1.1: Definition of isomorphism of pairs.

By definition, two pairs (η, H) and $(\overline{\eta}, \overline{H})$ are isomorphic if there exists a $\mathrm{Spin}^{\mathbb{C}}(r)$-equivariant diffeomorphism $\Psi : \mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)} \to \overline{\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}}$ such that Diagram 1.1 commutes.

A **Spin$^{\mathbb{C}}$ structure** on the vector bundle $E \to W$ is a pair (η, H) up to isomorphism.

The set of all Spin$^{\mathbb{C}}$ structures on the vector bundle $E \to W$ is indicated with $\mathrm{Spin}^{\mathbb{C}}(E \to W)$.

A **Spin$^{\mathbb{C}}$ structure** on an oriented manifold W is Spin$^{\mathbb{C}}$ structure on the tangent bundle $TW \to W$. In this case $r = n$ and the associated $\mathrm{SO}(n)$-principal bundle is the bundle of orthonormal frames (cf. Example 1.29).

The set of all Spin$^{\mathbb{C}}$ structures on W is indicated with $\mathrm{Spin}^{\mathbb{C}}(W)$.

Lemma 1.38. *Let $(\eta : \mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)} \to W, H)$ be a pair defining a Spin$^{\mathbb{C}}$ structure on $E \to W$, and let $(U_\alpha)_{\alpha \in A}$ be a fixed acyclic cover of W. By Lemma 1.34, there exist a cocycle $(h_{\alpha\beta})$ in $\check{\mathrm{C}}^1(W; C^\infty \mathrm{Spin}^{\mathbb{C}}(r))$ that represents $\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}$ and a cocycle $(g_{\alpha\beta})$ in $\check{\mathrm{C}}^1(W; C^\infty \mathrm{SO}(r))$ that represents $\mathcal{P}_{\mathrm{SO}(r)}$ (the $\mathrm{SO}(r)$-principal bundle associated to $E \to W$).*

Then, up to isomorphism of the pair (η, H), $(h_{\alpha\beta})$ is a lift of $(g_{\alpha\beta})$ to $\mathrm{Spin}^{\mathbb{C}}(r)$, i.e. $\forall\, \alpha, \beta$

$$\sigma(h_{\alpha\beta}) = g_{\alpha\beta}.$$

Proof. As $\mathcal{P}_{\mathrm{Spin}^{\mathbb{C}}(r)}$ furnishes a Spin$^{\mathbb{C}}$ structure on $E \to W$, $(\sigma(h_{\alpha\beta}))$ must describe the $\mathrm{SO}(r)$-principal bundle $\mathcal{P}_{\mathrm{SO}(r)}$ (up to isomorphism). Thus, there exists a family of C^∞ maps $(f_\alpha : U_\alpha \to \mathrm{SO}(r))$ such that $\forall\, \alpha, \beta$

$$f_\alpha\, \sigma(h_{\alpha\beta})\, f_\beta^{-1} = g_{\alpha\beta}.$$

Choose arbitrary lifts $\widetilde{f}_\alpha : U_\alpha \to \mathrm{Spin}^{\mathbb{C}}(r)$. Let $(\widetilde{g}_{\alpha\beta})$ be the $\mathrm{Spin}^{\mathbb{C}}(r)$ 1-cocycle defined by

$$\widetilde{g}_{\alpha\beta} = \widetilde{f}_\alpha\, h_{\alpha\beta}\, \widetilde{f}_\beta^{-1}.$$

As σ is a homomorphism, the $\mathrm{Spin}^{\mathbb{C}}(r)$ cocycle $(\widetilde{g}_{\alpha\beta})$ projects exactly on $g_{\alpha\beta}$. Moreover, the principal bundle associated to $(\widetilde{g}_{\alpha\beta})$ defines the same $\mathrm{Spin}^{\mathbb{C}}$ structure as η (the diffeomorphism Ψ is obtained by using the functions $\widetilde{f}_\alpha$). Hence the Lemma is proved. $\square$

A consequence of Lemma 1.38 is the following Corollary.

Corollary 1.39. *If $E \to W$ is an oriented real vector bundle and (U_α) is a fixed acyclic cover of W, then every Spin$^{\mathbb{C}}$ structure on $E \to W$ can be represented as a* $\mathrm{Spin}^{\mathbb{C}}(r)$*-cocycle $(\widetilde{g}_{\alpha\beta})$ that projects on the cocycle $(g_{\alpha\beta})$ defining $\mathcal{P}_{\mathrm{SO}(r)}$ (the* $\mathrm{SO}(r)$*-principal bundle associated to $E \to W$):*

$$\sigma(\widetilde{g}_{\alpha\beta}) = g_{\alpha\beta}.$$

Lemma 1.40. *Let $(\widetilde{g}_{\alpha\beta})$ and $(\widetilde{g}'_{\alpha\beta})$ be two* $\mathrm{Spin}^{\mathbb{C}}(r)$*-cocycles representing two Spin$^{\mathbb{C}}$ structures on $E \to W$ as in Corollary 1.39.*

The two Spin$^{\mathbb{C}}$ structures are the same if and only if there exist C^∞ functions $(f_\alpha : U_\alpha \to S^1)$ such that

$$\widetilde{g}'_{\alpha\beta} = f_\alpha^{-1}\, f_\beta\, \widetilde{g}_{\alpha\beta}. \tag{1.4}$$

It is noteworthy that S^1 is here thought of as a subgroup of $\mathrm{Spin}^{\mathbb{C}}(r)$:

$$S^1 = \mathrm{U}(1) = \ker\sigma = \frac{\mathrm{U}(1) \times \{e_{\mathrm{Spin}(r)}\}}{\{(1,1),(-1,-1)\}} \subseteq \mathrm{Spin}^{\mathbb{C}}(r).$$

Such a subgroup is also the centre of $\mathrm{Spin}^{\mathbb{C}}(r)$ for $r > 2$ (whereas $\mathrm{Spin}^{\mathbb{C}}(2)$ is abelian).

Proof of Lemma 1.40. By Lemma 1.34, $(\widetilde{g}_{\alpha\beta})$ and $(\widetilde{g}'_{\alpha\beta})$ define isomorphic $\mathrm{Spin}^{\mathbb{C}}(r)$-principal bundles if and only if there exists a collection of C^∞ maps $(f_\alpha : U_\alpha \to \mathrm{Spin}^{\mathbb{C}}(r))$ such that

$$\widetilde{g}'_{\alpha\beta} = f_\alpha\, g_{\alpha\beta}\, f_\beta^{-1.} \tag{1.5}$$

Since $(\widetilde{g}_{\alpha\beta})$ and $(\widetilde{g}'_{\alpha\beta})$ are chosen as in Corollary 1.39, both of them project on $g_{\alpha\beta}$, the cocycle defining $\mathcal{P}_{\mathrm{SO}(r)}$, the $\mathrm{SO}(r)$-principal bundle associated to $E \to W$. The fact that the two cocycles define the same $\mathrm{Spin}^{\mathbb{C}}$ structure is equivalent to the commutativity of Diagram 1.1. As the map Ψ is locally given by left multiplication by f_α, Diagram 1.1 locally becomes the following diagram:

$$\begin{tikzcd}
U_\alpha \times \mathrm{Spin}^{\mathbb{C}}(r) \arrow[rr, "\mathrm{id} \times f_\alpha \cdot"] \arrow[dr, "\mathrm{id} \times \sigma"'] & & U_\alpha \times \mathrm{Spin}^{\mathbb{C}}(r) \arrow[dl, "\mathrm{id} \times \sigma"] \\
& U_\alpha \times \mathrm{SO}(r) &
\end{tikzcd}$$

$$\begin{tikzcd}
(x, q) \arrow[rr, mapsto, "\mathrm{id} \times f_\alpha \cdot"] \arrow[dr, mapsto, "\mathrm{id} \times \sigma"'] & & (x, f_\alpha(x) \cdot q) \arrow[dl, mapsto, "\mathrm{id} \times \sigma"] \\
& (x, \sigma(q)) &
\end{tikzcd}$$

Hence the commutativity of the diagram is equivalent to the local equation:

$$\forall\, x \in U_\alpha,\, q \in \mathrm{Spin}^{\mathbb{C}}(r) \quad \sigma(q) = \sigma(f_\alpha(x)\, q). \tag{1.6}$$

As σ is a homomorphism, Equation (1.6) becomes

$$\forall\, x \in U_\alpha,\, q \in \mathrm{Spin}^{\mathbb{C}}(r) \quad \sigma(q) = \sigma(f_\alpha(x))\, \sigma(q),$$

which is in turn equivalent to $\sigma(f_\alpha(x)) = 1$ and so to the fact that the image of f_α is contained in $\ker \sigma = \mathrm{U}(1) = S^1$.

Thus, the cocycles $\widetilde{g}_{\alpha\beta}$ and $\widetilde{g}'_{\alpha\beta}$ define the same Spin$^{\mathbb{C}}$ structure if and only if Equation (1.5) holds for some f_α with image in S^1, hence (since S^1 is in the centre of Spin$^{\mathbb{C}}(r)$), if and only if Equation (1.4) holds. $\square$

We will see that the set of Spin$^{\mathbb{C}}$ structures on a vector bundle $E \to W$ is endowed with a free and transitive left action of $\mathrm{H}^2(W; \mathbb{Z})$, therefore (if non-empty) it is an $\mathrm{H}^2(W; \mathbb{Z})$-affine space.

Lemma 1.41. *The short exponential exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\exp} S^1 \longrightarrow 0.$$

gives natural isomorphisms

$$\check{\mathrm{H}}^k(W; C^\infty S^1) \cong \mathrm{H}^{k+1}(W; \mathbb{Z})$$

for each k.

Proof. Consider the associated long exact sequence in cohomology. All the groups $\check{\mathrm{H}}^*(W; C^\infty\mathbb{R})$ vanish due to the existence of a partition of unity (ρ_α). Indeed the map

$$(k(\varphi))_{\alpha_0 \ldots \alpha_{p-1}} = \sum_\alpha \rho_\alpha \varphi_{\alpha\alpha_0\ldots\alpha_{p-1}}$$

satisfies $k\delta + \delta k = \mathrm{id}$ and hence gives a homotopy between the maps id and 0 on the complex $\check{\mathrm{C}}^*(W; C^\infty\mathbb{R})$. Thus, the long exact sequence in cohomology gives the desired isomorphism. $\square$

Remark. In particular, there is an isomorphism

$$\check{\mathrm{H}}^1(W; C^\infty S^1) \cong \mathrm{H}^2(W; \mathbb{Z}).$$

Remark. It is noteworthy that, if $(\xi_{\alpha\beta})$ corresponds to ξ and $(\eta_{\alpha\beta})$ corresponds to η, the S^1-cocycle corresponding to $\xi + \eta$ is $(\xi_{\alpha\beta}\,\eta_{\alpha\beta})$ (this happens because exp carries sums to products).

Definition 1.42. Let W be an n-manifold, and (U_α) an acyclic cover of W. Let $E \to W$ be an oriented real vector bundle, $\mathcal{P}_{\mathrm{SO}(r)}$ its associated $\mathrm{SO}(r)$-principal bundle, and $(g_{\alpha\beta}) \in \check{\mathrm{Z}}^1(W; C^\infty \mathrm{SO}(r))$ the cocycle representing $\mathcal{P}_{\mathrm{SO}(r)}$.

The set of cocycles $(\widetilde{g}_{\alpha\beta}) \in \check{\mathrm{Z}}^1(W; C^\infty \mathrm{Spin}^{\mathbb{C}}(r))$ that project on $(g_{\alpha\beta})$ (i.e. such that $\sigma(\widetilde{g}_{\alpha\beta}) = g_{\alpha\beta}$) will be denoted by

$$\check{\mathrm{Z}}^1_{(g_{\alpha\beta})}(W; C^\infty \mathrm{SO}(r)).$$

Theorem 1.43. *Let W be an n-manifold, and (U_α) an acyclic cover of W. Let $E \to W$ be a real vector bundle, $\mathcal{P}_{\mathrm{SO}(r)}$ its associated $\mathrm{SO}(r)$-principal bundle, and $(g_{\alpha\beta}) \in \check{\mathrm{Z}}^1(W; C^\infty \mathrm{SO}(r))$ the cocycle representing $\mathcal{P}_{\mathrm{SO}(r)}$. For each $[\xi] \in \mathrm{H}^2(W; \mathbb{Z})$, let $(\xi_{\alpha\beta})$ denote the corresponding cohomology class given by the isomorphism $\mathrm{H}^2(W; \mathbb{Z}) \cong \check{\mathrm{H}}^1(W; C^\infty S^1)$.*

The map

$$\begin{aligned} \mathrm{H}^2(W; \mathbb{Z}) \times \check{\mathrm{Z}}^1_{(g_{\alpha\beta})}(W; C^\infty \mathrm{Spin}^{\mathbb{C}}(r)) &\longrightarrow \check{\mathrm{Z}}^1_{(g_{\alpha\beta})}(W; C^\infty \mathrm{Spin}^{\mathbb{C}}(r)) \\ (\xi, (\widetilde{g}_{\alpha\beta})) &\longmapsto (\widetilde{g}_{\alpha\beta}\,\xi_{\alpha\beta}) \end{aligned}$$

induces a free and transitive action of $\mathrm{H}^2(W; \mathbb{Z})$ on $\mathrm{Spin}^{\mathbb{C}}(E \to W)$.

Proof. Two cocycles define the same $\mathrm{Spin}^{\mathbb{C}}$ structure if and only if Equation (1.4) holds, so equivalent cocycles are mapped to equivalent cocycles (everything commutes as $U(1) = S^1$ is in the centre of $\mathrm{Spin}^{\mathbb{C}}(r)$), and, thus, there is an induced map

$$\mathrm{H}^2(W; \mathbb{Z}) \times \mathrm{Spin}^{\mathbb{C}}(E \to W) \longrightarrow \mathrm{Spin}^{\mathbb{C}}(E \to W),$$

which is clearly an action of $\mathrm{H}^2(W; \mathbb{Z})$ on $\mathrm{Spin}^{\mathbb{C}}(E \to W)$.

We have now to check that the action is free and transitive. To check that it is free, suppose that there exist a $\mathrm{Spin}^{\mathbb{C}}(r)$-cocycle $(\widetilde{g}_{\alpha\beta})$ and a S^1-cocycle $(\xi_{\alpha\beta})$ such that $(\widetilde{g}_{\alpha\beta}\,\xi_{\alpha\beta})$ gives the same $\mathrm{Spin}^{\mathbb{C}}$ structure as $(\widetilde{g}_{\alpha\beta})$. Then, by Lemma 1.40, there exist C^∞ functions $(f_\alpha : U_\alpha \to S^1 = \mathrm{U}(1))$ such that

$$\widetilde{g}_{\alpha\beta}\,\xi_{\alpha\beta} = f_\alpha^{-1}\,f_\beta\,\widetilde{g}_{\alpha\beta}.$$

As $\xi_{\alpha\beta}$, f_α^{-1} and f_β lie in the centre of $\mathrm{Spin}^{\mathbb{C}}(r)$, the previous equation becomes (upon multiplication by $\widetilde{g}_{\alpha\beta}^{-1}$)

$$\xi_{\alpha\beta} = f_\alpha^{-1}\,f_\beta.$$

This implies that $(\xi_{\alpha\beta})$ is a coboundary, so the action is free.

To check the transitivity of the action, let $(\widetilde{g}_{\alpha\beta})$ and $(\widetilde{g}'_{\alpha\beta})$ be two cocycles in $\check{\mathrm{Z}}^1_{(g_{\alpha\beta})}(W; C^\infty \operatorname{Spin}^{\mathbb{C}}(r))$. Choose lifts $(z_{\alpha\beta}, h_{\alpha\beta})$ and $(z'_{\alpha\beta}, h'_{\alpha\beta})$ of $(\widetilde{g}_{\alpha\beta})$ and $(\widetilde{g}'_{\alpha\beta})$ to $\mathrm{U}(1)\times\mathrm{Spin}(r)$. Multiplying $h'_{\alpha\beta}$ and $z'_{\alpha\beta}$ by -1 where necessary, we may assume that $h_{\alpha\beta} = h'_{\alpha\beta}$.

Let $\xi_{\alpha\beta} = z'_{\alpha\beta}/z_{\alpha\beta}$. If $(\xi_{\alpha\beta})$ is a cocycle, this would prove the transitivity of the action, since multiplication by $\xi_{\alpha\beta}$ clearly sends $(\widetilde{g}_{\alpha\beta})$ to $(\widetilde{g}'_{\alpha\beta})$. Hence the only thing to prove is that $(\xi_{\alpha\beta})$ is a cocycle.

Since $(\widetilde{g}_{\alpha\beta})$ and $(\widetilde{g}'_{\alpha\beta})$ are cocycles,

$$\widetilde{g}_{\alpha\beta}\,\widetilde{g}_{\beta\gamma}\,\widetilde{g}_{\gamma\alpha} = 1,$$

$$\widetilde{g}'_{\alpha\beta}\,\widetilde{g}'_{\beta\gamma}\,\widetilde{g}'_{\gamma\alpha} = 1.$$

These relations in $\mathrm{U}(1) \times \mathrm{Spin}(r)$ become

$$(z_{\alpha\beta}\,z_{\beta\gamma}\,z_{\gamma\alpha}, h_{\alpha\beta}\,h_{\beta\gamma}\,h_{\gamma\alpha}) \in \{(1,1), (-1,-1)\},$$

$$(z'_{\alpha\beta}\,z'_{\beta\gamma}\,z'_{\gamma\alpha}, h_{\alpha\beta}\,h_{\beta\gamma}\,h_{\gamma\alpha}) \in \{(1,1), (-1,-1)\}.$$

There are two possible cases. Either

$$z_{\alpha\beta}\,z_{\beta\gamma}\,z_{\gamma\alpha} = h_{\alpha\beta}\,h_{\beta\gamma}\,h_{\gamma\alpha} = z'_{\alpha\beta}\,z'_{\beta\gamma}\,z'_{\gamma\alpha} = 1$$

or

$$z_{\alpha\beta}\,z_{\beta\gamma}\,z_{\gamma\alpha} = h_{\alpha\beta}\,h_{\beta\gamma}\,h_{\gamma\alpha} = z'_{\alpha\beta}\,z'_{\beta\gamma}\,z'_{\gamma\alpha} = -1.$$

In any case

$$\xi_{\alpha\beta}\,\xi_{\beta\gamma}\,\xi_{\gamma\alpha} = \frac{z'_{\alpha\beta}\,z'_{\beta\gamma}\,z'_{\gamma\alpha}}{z_{\alpha\beta}\,z_{\beta\gamma}\,z_{\gamma\alpha}} = 1,$$

hence $(\xi_{\alpha\beta})$ is a cocycle. □

Theorem 1.44. *Let W be a manifold (possibly with non-empty boundary) such that $\mathrm{H}^3(W;\mathbb{Z})$ has no 2-torsion.*

Then there exists a Spin$^{\mathbb{C}}$ structure on every oriented real vector bundle $E \to W$ with rank $r \geq 2$.

Proof. Let (U_α) be an acyclic cover of W, and let $(g_{\alpha\beta})$ be the $\mathrm{SO}(r)$-cocycle representing E. Let $h_{\alpha\beta}$ arbitrary lifts of $g_{\alpha\beta}$ to $\mathrm{Spin}(r)$.

Consider the exact sequence

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(r) \longrightarrow \mathrm{SO}(r) \longrightarrow 0.$$

The long exact sequence in cohomology is

$$\ldots \to \check{\mathrm{H}}^1(W; C^\infty \operatorname{Spin}(r)) \longrightarrow \check{\mathrm{H}}^1(W; C^\infty \operatorname{SO}(r)) \xrightarrow{w_2} \check{\mathrm{H}}^2(W; \{\pm 1\}) \to \ldots$$

where the boundary map w_2 is the assignment of the second Stiefel-Whitney class. The definition of the boundary map of the long exact sequence in cohomology implies that the cocycle

$$w_{\alpha\beta\gamma} = h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha} \in \check{\mathrm{C}}^2(W;\{\pm 1\})$$

represents the second Stiefel-Whitney class $w_2(E)$ (under the identification $(\mathbb{Z}_2,+) \stackrel{(-1)^{\cdot}}{\longrightarrow} (\{\pm 1\},\cdot)$).

The long exact sequence in cohomology

$$\ldots \longrightarrow \check{\mathrm{H}}^2(W;\mathbb{Z}) \longrightarrow \check{\mathrm{H}}^2(W;\mathbb{Z}_2) \longrightarrow \check{\mathrm{H}}^3(W;\mathbb{Z}) \xrightarrow{2\cdot} \check{\mathrm{H}}^3(W;\mathbb{Z}) \longrightarrow \ldots$$

together with the fact that $\mathrm{H}^3(W;\mathbb{Z})$ has no 2-torsion implies that there exists a lift $c \in \mathrm{H}^2(W;\mathbb{Z})$ of $w_2(E) \in \mathrm{H}^2(W;\mathbb{Z}_2)$.

Now recall (cf. Lemma 1.41) that the exponential exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \stackrel{\exp}{\longrightarrow} S^1 \longrightarrow 0$$

yields an isomorphism

$$\check{\mathrm{H}}^1(W;C^\infty S^1) \cong \mathrm{H}^2(W;\mathbb{Z}),$$

and so there exists a cocycle $(l_{\alpha\beta}) \in \check{\mathrm{C}}^1(W;C^\infty S^1)$ representing c. Let $\widetilde{l}_{\alpha\beta}$ be arbitrary lifts to $\mathbb{R}$ of $l_{\alpha\beta}$. The definition of the boundary map implies that

$$[c] = \left[\widetilde{l}_{\alpha\beta} + \widetilde{l}_{\beta\gamma} + \widetilde{l}_{\gamma\alpha}\right] \in \mathrm{H}^2(W;\mathbb{Z}). \tag{1.7}$$

Applying the composition map

$$\mathrm{H}^2(W;\mathbb{Z}) \xrightarrow{(\mathrm{mod}\ 2)} \mathrm{H}^2(W;\mathbb{Z}_2) \xrightarrow{(-1)^{\cdot}} \check{\mathrm{H}}^2(W;\{\pm 1\})$$

Equation (1.7) implies

$$[(w_{\alpha\beta\gamma})] = [(l_{\alpha\beta}\, l_{\beta\gamma}\, l_{\gamma\alpha})]\,.$$

Since $(w_{\alpha\beta\gamma})_{\alpha\beta\gamma}$ and $(l_{\alpha\beta}\, l_{\beta\gamma}\, l_{\gamma\alpha})_{\alpha\beta\gamma}$ represent the same cohomology class in $\check{\mathrm{H}}^2(W;\{\pm 1\})$, there exists a 1-cocycle $(\varepsilon_{\alpha\beta}) \in \check{\mathrm{H}}^1(W;\{\pm 1\})$ such that

$$l_{\alpha\beta}\, l_{\beta\gamma}\, l_{\gamma\alpha}\, \varepsilon_{\alpha\beta}\, \varepsilon_{\beta\gamma}\, \varepsilon_{\gamma\alpha} = w_{\alpha\beta\gamma}.$$

Define $l'_{\alpha\beta} = l_{\alpha\beta}\, \varepsilon_{\alpha\beta}$. Now the cochain $\left[l'_{\alpha\beta}, h_{\alpha\beta}\right]$ is a $\mathrm{Spin}^{\mathbb{C}}(r)$-cocycle because

$$\left[l'_{\alpha\beta}, h_{\alpha\beta}\right] \cdot \left[l'_{\beta\gamma}, h_{\beta\gamma}\right] \cdot \left[l'_{\gamma\alpha}, h_{\gamma\alpha}\right] = \left[w_{\alpha\beta\gamma}, w_{\alpha\beta\gamma}\right],$$

and therefore it defines a $\mathrm{Spin}^{\mathbb{C}}$ structure on E. □

Corollary 1.45. *Let W be an n-dimensional manifold with $\mathrm{H}^3(W;\mathbb{Z})$ without 2-torsion, and let $E \to W$ an oriented real vector bundle.*

Then $\mathrm{Spin}^{\mathbb{C}}(E \to W)$ is an affine space over $\mathrm{H}^2(W;\mathbb{Z})$.

Proof. The statement follows directly from Theorem 1.43 and Theorem 1.44. □

1.3.3 Restriction map

If W is an n-dimensional oriented manifold (with $n \geq 3$), it is possible to define a restriction map from $\mathrm{Spin}^{\mathbb{C}}(W)$ to $\mathrm{Spin}^{\mathbb{C}}(\partial W)$. The aim of the current subsection is to define this map.

First, note that there exists a lift j of the standard inclusion $i : \mathrm{SO}(r) \to \mathrm{SO}(r+1)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spin}^{\mathbb{C}}(r) \arrow[r, "j"] \arrow[d, "\sigma_r"] & \mathrm{Spin}^{\mathbb{C}}(r+1) \arrow[d, "\sigma_{r+1}"] \\ \mathrm{SO}(r) \arrow[r, "i"'] & \mathrm{SO}(r+1) \end{tikzcd}$$

Lemma 1.46. *Let $E \to W$ be an oriented real vector bundle. Then, there exists a canonical $H^2(W;\mathbb{Z})$-equivariant bijection*

$$\Psi_E : \mathrm{Spin}^{\mathbb{C}}(E \to W) \xrightarrow{\sim} \mathrm{Spin}^{\mathbb{C}}(E \oplus \mathbb{R} \to W),$$

which maps the Spin$^{\mathbb{C}}$ structure on $E \to W$ defined by the cocycle $(\widetilde{g}_{\alpha\beta})$ to the Spin$^{\mathbb{C}}$ structure on $E \oplus \mathbb{R} \to W$ defined by the cocycle $j(\widetilde{g}_{\alpha\beta})$.

Proof. $j(\widetilde{g}_{\alpha\beta})$ represents a Spin$^{\mathbb{C}}$ structure on $E \oplus \mathbb{R} \to W$ because the cocycle $\sigma_{r+1}(j(\widetilde{g}_{\alpha\beta})) = i(g_{\alpha\beta})$ is the cocycle representing $E \oplus \mathbb{R} \to W$.

The map is well defined because $j : \mathrm{Spin}^{\mathbb{C}}(r) \supseteq S^1 \to S^1 \subseteq \mathrm{Spin}^{\mathbb{C}}(r+1)$ acts as the identity, so equivalent cocycles (cf. Lemma 1.40) are sent to equivalent cocycles:

$$j(f_\alpha^{-1}\, f_\beta\, \widetilde{g}_{\alpha\beta}) = f_\alpha^{-1}\, f_\beta\, j(\widetilde{g}_{\alpha\beta}).$$

Moreover, as $j_{|S^1}$ acts as the identity, it is clear that the map

$$\mathrm{Spin}^{\mathbb{C}}(E \to W) \to \mathrm{Spin}^{\mathbb{C}}(E \oplus \mathbb{R} \to W)$$

is $\mathrm{H}^2(W;\mathbb{Z})$-equivariant:

$$j(\xi_{\alpha\beta}\, \widetilde{g}_{\alpha\beta}) = \xi_{\alpha\beta}\, j(\widetilde{g}_{\alpha\beta}).$$

The fact that j is bijective follows at once from the $\mathrm{H}^2(W;\mathbb{Z})$-equivariance. $\square$

Let W be an oriented n-manifold with boundary ∂W. There is a clear identification $TW_{|\partial W} = \mathbb{R} \oplus T(\partial W)$, given by the 'outer normal first' rule.

Theorem 1.47. *Let W be an oriented n-manifold with boundary ∂W, with $n \geq 3$. The composition map*

$$\begin{array}{ccccc} \mathrm{Spin}^{\mathbb{C}}(W) & \xrightarrow{r_\partial} & \mathrm{Spin}^{\mathbb{C}}(TW_{|\partial W} \to \partial W) & & \\ & & \| & & \\ & & \mathrm{Spin}^{\mathbb{C}}(\mathbb{R} \oplus T(\partial W) \to \partial W) & \xrightarrow{\Psi^{-1}_{T(\partial W)}} & \mathrm{Spin}^{\mathbb{C}}(\partial W) \\ \eta & \longmapsto & \eta_{|(TW_{|\partial W})} & \longmapsto & \Psi^{-1}_{T(\partial W)}(\eta_{|(TW_{|\partial W})}) \end{array}$$

is equivariant for the action of $i^* : \mathrm{H}^2(W;\mathbb{Z}) \longrightarrow \mathrm{H}^2(\partial W;\mathbb{Z})$ *(the map induced by the inclusion* $i : \partial W \to W$*), i.e. it satisfies*

$$\Psi^{-1}_{T(\partial W)} \circ r_\partial(h \cdot \eta) = i^*(h) \cdot \Psi^{-1}_{T(\partial W)} \circ r_\partial(\eta).$$

$\Psi^{-1}_{T(\partial W)} \circ r_\partial$ *is called* ***restriction map****.*

Proof. Let $(\widetilde{g}_{\alpha\beta})$ is a cocycle representing a $\mathrm{Spin}^{\mathbb{C}}$ structure. The map r_∂ acts as follows:

$$r_\partial(\widetilde{g}_{\alpha\beta}) = (\widetilde{g}_{\alpha\beta|\partial W}).$$

As i^* is the restriction of the cocycles to ∂W, r_∂ is clearly equivariant for the action of i^*.

The second map, $\Psi^{-1}_{T(\partial W)}$, is $\mathrm{H}^2(\partial W;\mathbb{Z})$-equivariant by Lemma 1.46. The theorem is proved. □

1.3.4 $\mathrm{Spin}^{\mathbb{C}}$ structures on 4-manifolds

Definition 1.48. Let $\Lambda = (Z, f)$ be a lattice. A covector $\chi \in Z^*$ is **characteristic** if

$$\chi(v) = f(v,v) \quad \forall\, v \in Z.$$

The set of characteristic covectors is denoted by $\mathrm{Char}(\Lambda)$.

Let X be a 4-manifold (possibly with boundary) with $\mathrm{H}_1(X;\mathbb{Z}) = 0$. In this subsection $\Lambda = (Z, f)$ will denote the lattice $(\mathrm{H}_2(X;\mathbb{Z}), Q_X)$. Note that $Z^* \cong \mathrm{H}^2(X;\mathbb{Z})$ by the Universal Coefficient Theorem (cf. [Hat02, Theorem 3.2]).

Lemma 1.49. *Let* X *be a* 4*-manifold (with boundary) with* $\mathrm{H}_1(X;\mathbb{Z}) = 0$. $\mathrm{Char}(\Lambda)$ *is endowed with a free and transitive action of* Z^**, defined by*

$$\xi \cdot \chi = \chi + 2\xi.$$

Proof. Suppose that $\chi + 2\xi = \chi$. Then $2\xi = 0$. As $\mathrm{H}_1(X;\mathbb{Z}) = 0$, the Universal Coefficient Theorem implies that $\mathrm{H}^2(X;\mathbb{Z})$ is torsion-free, so $\xi = 0$ and the action is free.

Now let χ_1 and χ_2 be two characteristic classes. Then $\chi_1 - \chi_2 \equiv 0 \pmod 2$. The exactness of the sequence

$$\mathrm{H}^2(X;\mathbb{Z}) \xrightarrow{2\cdot} \mathrm{H}^2(X;\mathbb{Z}) \xrightarrow{(\mathrm{mod}\ 2)} \mathrm{H}^2(X;\mathbb{Z}_2)$$

implies that $\chi_1 - \chi_2 = 2\xi$ for some $\xi \in \mathrm{H}^2(X;\mathbb{Z})$, so the action is transitive. $\square$

Definition 1.50. Let $E \to W$ be an an oriented real vector bundle, where W is an n-manifold (possibly with boundary). Let $(\widetilde{g}_{\alpha\beta}) = ([\lambda_{\alpha\beta}, h_{\alpha\beta}])$ represent a Spin$^{\mathbb{C}}$ structure on $E \to W$.

The **determinant bundle** associated to the Spin$^{\mathbb{C}}$ structure is the complex line bundle associated to the S^1-cocycle $(\lambda^2_{\alpha\beta})$.

Theorem 1.51. *Let X be an oriented 4-manifold (possibly with boundary) with $\mathrm{H}_1(X;\mathbb{Z}) = 0$ and $\mathrm{H}^3(X;\mathbb{Z}) = 0$ (note that by Poincaré duality the last condition is redundant if $\partial X = \emptyset$).*

Then, the map

$$\mathrm{Spin}^{\mathbb{C}}(X) \xrightarrow{c_1} \mathrm{Char}(\Lambda)$$

that maps a Spin$^{\mathbb{C}}$ structure to the first Chern class of its determinant bundle is Z^*-equivariant and, thus, it is a bijection.

Proof. The short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}^{\mathbb{C}}(4) \to \mathrm{U}(1) \times \mathrm{SO}(4) \to 0$$

gives a long exact sequence in cohomology

$$\begin{aligned} \dots \longrightarrow \check{\mathrm{H}}^1(X; C^\infty \, \mathrm{Spin}^{\mathbb{C}}(4)) &\longrightarrow \check{\mathrm{H}}^1(X; C^\infty \, \mathrm{U}(1)) \oplus \check{\mathrm{H}}^1(X; C^\infty \, \mathrm{SO}(4)) \\ \xrightarrow{\partial \,=\, c_1 + w_2} \check{\mathrm{H}}^2(X;\mathbb{Z}_2) &\longrightarrow \dots \end{aligned}$$

where the boundary map ∂ is the modulo 2 sum of the first Chern class and the second Stiefel-Whitney class:

$$\partial((\lambda_{\alpha\beta}), (g_{\alpha\beta})) \equiv c_1(\lambda_{\alpha\beta}) + w_2(g_{\alpha\beta}) \pmod 2.$$

If $\mathfrak{s}$ is a Spin$^{\mathbb{C}}$ structure on X, by exactness

$$c_1(\mathfrak{s}) + w_2(TX) \equiv 0 \pmod 2,$$

so, if $\overline{v}$ is the modulo 2 class of v,

$$\begin{aligned} \langle c_1(\mathfrak{s}), v\rangle &\equiv \langle w_2(TX), \overline{v}\rangle \pmod 2 \\ &\equiv f(v,v) \pmod 2, \end{aligned}$$

where the second equality is Wu's formula (cf. [Sco05, Ch. 4, Sect. 4.3]). Hence the image of a Spin$^{\mathbb{C}}$ structure is a characteristic class.

Now let $\mathfrak{s} = ([\lambda_{\alpha\beta}, h_{\alpha\beta}])$ be a Spin$^{\mathbb{C}}$ structure and $\xi = (\xi_{\alpha\beta}) \in Z^*$ (recall that, by Lemma 1.40, $Z^* = \mathrm{H}^2(X;\mathbb{Z}) = \check{\mathrm{H}}(X; C^\infty S^1)$).

$$\begin{aligned} c_1(\xi \cdot \mathfrak{s}) &= c_1([\lambda_{\alpha\beta}\xi_{\alpha\beta}, h_{\alpha\beta}]) \\ &= c_1(\lambda^2_{\alpha\beta}\xi^2_{\alpha\beta}) \\ &= c_1(\lambda^2_{\alpha\beta}) + 2c_1(\xi_{\alpha\beta}) \\ &= c_1(\mathfrak{s}) + 2\xi, \end{aligned}$$

so the map $c_1 : \mathrm{Spin}^{\mathbb{C}}(X) \to \mathrm{Char}(\Lambda)$ is Z^*-equivariant.

As $\mathrm{Spin}^{\mathbb{C}}(X)$ and $\mathrm{Char}(\Lambda)$ are Z^*-affine spaces and c_1 is a Z^*-equivariant map, c_1 is also a bijection. $\square$

Let X be a compact 4-manifold with boundary $\partial X = Y$, such that $\mathrm{H}_1(X;\mathbb{Z}) = 0$. Let $\Lambda = (Z, f)$ be the lattice $(\mathrm{H}_2(X;\mathbb{Z}), Q_X)$, and let

$$\begin{aligned} \widehat{f} : Z &\longrightarrow Z^* \\ \lambda &\longmapsto f(\lambda, \cdot) \end{aligned}$$

be the function adjoint to f.

Consider Diagram 1.2 (where the coefficients are omitted because they are all $\mathbb{Z}$). $\mathrm{H}^3(X, Y;\mathbb{Z}) \cong \mathrm{H}_1(X;\mathbb{Z}) = 0$ by Poincaré-Lefschetz duality. The map ev is an isomorphism due to the Universal Coefficient Theorem.

Hence $\widehat{f}$ is a presentation map for $\mathrm{H}^2(Y;\mathbb{Z})$, i.e.

$$\mathrm{H}^2(Y;\mathbb{Z}) \cong Z^*/\widehat{f}(Z).$$

$$\begin{tikzcd} \mathrm{H}^2(X,Y) \arrow[r] \arrow[d, "\mathrm{PD}", "\wr"'] & \mathrm{H}^2(X) \arrow[r] \arrow[d, "\mathrm{ev}", "\wr"'] & \mathrm{H}^2(Y) \arrow[r] & \mathrm{H}^3(X,Y) \arrow[d, equal] \\ Z \arrow[r, dashed, "\widehat{f}"] & Z^* & & 0 \end{tikzcd}$$

Diagram 1.2: The long exact sequence in cohomology with $\mathbb{Z}$ coefficients for a pair (X^4, Y^3) with $\mathrm{H}_1(X) = 0$.

Lemma 1.52. *Let X be a compact 4-manifold with boundary $\partial X = Y$, such that $\mathrm{H}_1(X;\mathbb{Z}) = 0$. Let $\Lambda = (Z, f)$ be the lattice $(\mathrm{H}_2(X;\mathbb{Z}), Q_X)$, and let $\widehat{f}$ denote the adjoint function.*

Then $\mathrm{Char}(\Lambda)/(2 \cdot \widehat{f}(Z))$ is an affine space over $\mathrm{H}^2(Y;\mathbb{Z}) = Z^*/\widehat{f}(Z)$.

Proof. The statement follows at once from the fact that Z^* acts on $\mathrm{Char}(\Lambda)$ freely and transitively (cf. Lemma 1.49). $\square$

Theorem 1.53. *Let L be a framed link in S^3. Let (X_L, S^3_L) be the pair obtained by L-surgery from S^3. Let $\Lambda = (Z, f)$ be the lattice $(\mathrm{H}_2(X_L;\mathbb{Z}), Q_{X_L})$, and let $\widehat{f}$ denote the adjoint function.*

The map

$$\begin{aligned} \mathrm{Spin}^{\mathbb{C}}(S^3_L) &\longrightarrow \mathrm{Char}(\Lambda)\big/(2 \cdot \widehat{f}(Z)) \\ \mathfrak{t} &\longmapsto [c_1(\mathfrak{s})] \end{aligned}$$

sending a Spin$^{\mathbb{C}}$ structure $\mathfrak{t}$ to the equivalence class of the first Chern class of one of its extensions to X, is an $\mathrm{H}^2(S^3_L;\mathbb{Z})$-equivariant bijection.

Corollary 1.54. *Under the assumptions of of Theorem 1.53, an extension of a Spin$^{\mathbb{C}}$ structure $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(S^3_L)$ to $\mathrm{Spin}^{\mathbb{C}}(X_L)$ always exists.*

Proof of Theorem 1.53. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Spin}^{\mathbb{C}}(X_L) \arrow[r, "Z^* \to Z^*/\widehat{f}(Z)", "R"'] \arrow[d, "c_1", "\wr"'] & \mathrm{Spin}^{\mathbb{C}}(S^3_L) \\
\mathrm{Char}(\Lambda) \arrow[ur, dashed, bend right, "\widetilde{R}"'] &
\end{tikzcd}$$

c_1 is a Z^*-equivariant bijection (cf. Theorem 1.51), and R is equivariant over $Z^* \to Z^*/\widehat{f}(Z)$ (cf. Theorem 1.47).

Hence $\widetilde{R}$ is equivariant for the action of $Z^* \to Z^*/\widehat{f}(Z)$ too and so it induces a $\left(Z^*/\widehat{f}(Z)\right)$-equivariant map on the quotient:

$$\mathrm{Char}(\Lambda)\Big/(2\cdot\widehat{f}(Z)) \to \mathrm{Spin}^{\mathbb{C}}(S^3_L).$$

The bijectivity of this map follows from the $\left(Z^*/\widehat{f}(Z)\right)$-equivariance and from the fact that both $\mathrm{Char}(\Lambda)/(2\cdot\widehat{f}(Z))$ and $\mathrm{Spin}^{\mathbb{C}}(S^3_L)$ are affine spaces on $Z^*/\widehat{f}(Z) = H^2(S^3_L;\mathbb{Z})$ (cf. Lemma 1.52 and Theorem 1.43). □

1.4 Branched covers

In this section a more general notion of covering map between manifolds is defined. This notion, that will be useful in the study of links in S^3, is the one of 'branched cover', which is fully explained in [Pie93].

The 'local model' to keep in mind is the following: let D^1 be the unit open disk in the complex field $\mathbb{C}$. Consider the map

$$\begin{aligned} p_r : & D^1 \to D^1 \\ & x \mapsto x^r \end{aligned}$$

Now, let W be an n-dimensional manifold (in our case n will always be 3). The 'local model' for n-dimensional manifolds becomes

$$\begin{aligned} p_r : & D^1 \times I^{n-2} \to D^1 \times I^{n-2} \\ & (x, \underline{t}) \mapsto (x^r, \underline{t}), \end{aligned}$$

where $I = (-1, 1)$.

Note that the map p_r is an r-sheets covering on the open subspace of $D^1 \times I^{n-2}$ defined by $\{x \neq 0\}$, and it is a homeomorphism on the closed complementary subspace. The general definition of branched cover is the following.

Definition 1.55. Let $\pi : \widetilde{W} \to W$ be a C^∞ application between n-manifolds with finite fibers.

π is a **branched cover** if $\forall \widetilde{x} \in \overset{\circ}{\widetilde{W}}$ (respectively $\partial\widetilde{W}$) there exist a neighbourhood $\widetilde{U}$ of $\widetilde{x}$ and a neighbourhood U of $x = \pi(\widetilde{x})$ such that the map $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ is diffeomorphic to $p_r : D^1 \times I^{n-2} \to D^1 \times I^{n-2}$ (respectively $p_r : D^1 \times [0,1)^{n-2} \to D^1 \times [0,1)^{n-2}$), that is there exist two diffeomorphisms $\widetilde{\psi} : \widetilde{U} \to D^1 \times I^{n-2}$ and $\psi : U \to D^1 \times I^{n-2}$ (resp. $\widetilde{\psi} : \widetilde{U} \to D^1 \times [0,1)^{n-2}$ and $\psi : U \to D^1 \times [0,1)^{n-2}$) such that $\widetilde{\psi}(\widetilde{x}) = (0,0)$ and the diagram on the left (respectively on the right) commutes:

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\cong", "\widetilde{\psi}"'] \arrow[d, "\pi"] & D^1 \times I^{n-2} \arrow[d, "p_r"] \\ U \arrow[r, "\cong", "\psi"'] & D^1 \times I^{n-2} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{U} \arrow[r, "\cong", "\widetilde{\psi}"'] \arrow[d, "\pi"] & D^1 \times [0,1)^{n-2} \arrow[d, "p_r"] \\ U \arrow[r, "\cong", "\psi"'] & D^1 \times [0,1)^{n-2} \end{tikzcd}$$

The number r is called **branching index** of π in $\widetilde{x}$, and it will be indicated by $\mathrm{ind}_\pi(\widetilde{x})$. The set $S_\pi = \{\widetilde{x} \in \widetilde{W} \mid \mathrm{ind}_\pi(\widetilde{x}) > 1\}$ is called **singular set** of π, and its elements are called **singular points**, whereas the set $B_\pi = \pi(S_\pi)$ is called **branch set**, and its elements are called **branch points**.

Moreover, if $x \in W$, a neighbourhood V of x is called **trivializing** if $\forall \widetilde{x} \in \pi^{-1}(x)$ there exist $\widetilde{U}_{\widetilde{x}}$ and $U_{\widetilde{x}}$ as above such that $U_{\widetilde{x}} \supseteq V$.

Remark. As a direct consequence of the definition, the singular set (resp. the branch set) is an $(n-2)$-dimensional submanifold properly embedded in $\widetilde{W}$ (resp. W), and the branching index is constant on the connected components of the singular set.

Remark. If $\pi : \widetilde{W} \to W$ is a branched cover whose branch set is B_π, then $\pi|_{\widetilde{W} \setminus \pi^{-1}(B_\pi)}$ is a (not branched) cover, which is called the **cover associated** to π.

Lemma 1.56 (Uniqueness of the branched cover)**.** *A branched cover of a given compact manifold W is completely determined by its branch set B_π and its associated cover*

Sketch of the proof. An explicit construction of the branched cover starting from B_π and the associated cover can be given. Let V be a tubular neighbourhood of the branch set, chosen in such a way that its closure is contained in the union of trivializing neighbourhoods of branch points (such a V exists due to the compactness of W). Now consider the associated cover, and restrict it to $\overline{W \setminus V}$. In order to get the branched cover, we have to glue $\pi^{-1}(V)$ to it. As $\pi^{-1}(V)$ is diffeomorphich to V, we may glue V to the restriction of the associated cover to let the cover branch, using the product structure $V \cong D^1 \times B_\pi$. □

1.4.1 Branched double covers

A particular family of branched covers will be very useful in this work. It is the family of branched double covers.

Let $L \subseteq S^3$ be a link, and let $\rho : \pi_1(S^3 \setminus L) \to \mathbb{Z}_2$ be the augmentation homomorphism, i.e. the homomorphism sending each meridian of L to 1. Note that ρ is well defined thanks to Wirtinger's theorem on the presentation of $\pi_1(S^3 \setminus L)$ (actually all the generators of Wirtinger's presentation are sent to 1, and the only relations in the presentation of $\pi_1(S^3 \setminus L)$ are conjugation relations between the generators). Moreover it is noteworthy that the definition of ρ does not depend on the choice of the orientation of each component of L.

Definition 1.57. If $L \subseteq S^3$ is an unoriented link, the double cover of S^3 branched along L (or, more simply, the **branched double cover** of L), denoted by $\Sigma(L)$, is the unique branched cover of S^3 whose branch set is L and whose associated cover is the cover of $S^3 \setminus L$ corresponding to $\ker \rho$.

Remark. The uniqueness of $\Sigma(L)$ is guaranteed by Lemma 1.56, whereas the existence is a consequence of the explicit construction.

$\Sigma(L)$ can be constructed explicitly as follows: let $\widetilde{C(L)}$ be the double cover of $C(L)$ corresponding to $\ker \rho$, and let $\widetilde{m}_1, \dots, \widetilde{m}_n$ be the lifts of 'double meridians' of each component of L (a 'double meridian' of a knot $K \subseteq S^3$ is a closed simple curve on $\partial N(K)$ such that in homology is twice a meridian). Note that $\widetilde{m}_1, \dots, \widetilde{m}_n$ are closed loops because the double meridians belong to $\ker \rho$. After attaching a solid torus to each component of the boundary of $\widetilde{C(L)}$ so that the meridian of the attaching torus is identified with $\widetilde{m}_i$, the final manifold will be $\Sigma(L)$.

Moreover, thanks to the product structure of the attaching solid tori, it is possible to extend the map $\pi : \widetilde{C(L)} \to C(L)$ to a map

$$\pi : \Sigma(L) \to S^3,$$

which is the double cover of S^3 branched along L.

The following result concerns the branched double cover of a link (for the definition of $\det L$ the reader may see [Lic97]).

Lemma 1.58 ([Lic97, Corollary 9.2])**.** *If $\Sigma(L)$ is the branched double cover of a link L, then*

$$\det L = |\,\mathrm{H}_1(\Sigma(L))|.$$

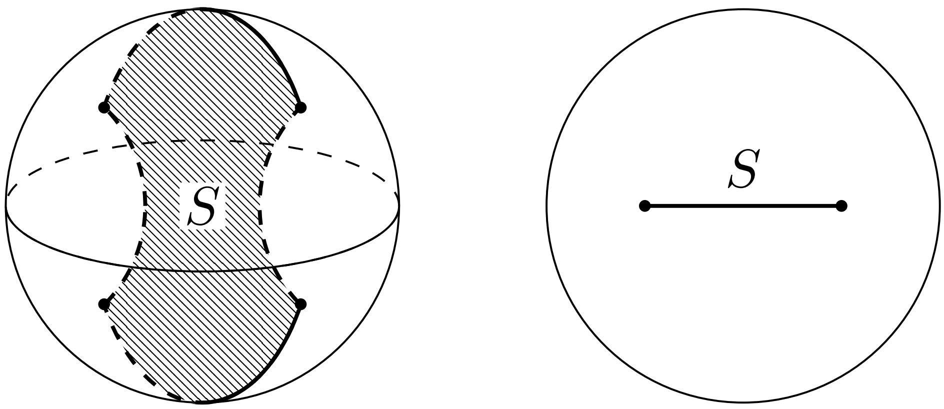


Figure 1.2: The picture on the left represents the 3-ball B with the two untangled arcs bounding the surface S. The picture on the right represents a horizontal section of B (the line is the section of the surface S). If the ball B is thought of as $B^2 \times I$, and the two arcs are $\{x\} \times I$ and $\{y\} \times I$, every (horizontal) section $B^2 \times \{t\}$ of B appears as in the picture on the right.

Example 1.59 (The double cover of a ball branched over two arcs). In Section 1.5 of this work we will need to know how the double cover of a ball branched along two untangled arcs looks like. Since in addition it is a clarifying example of what a branched cover is, it is fully explained in this example.

Let B be a 3-ball with two properly embedded untangled arcs a and b (recall that W_2 is properly embedded in W_1 if W_2 is embedded in W_1 and $\partial W_2 = \partial W_1 \cap W_2$). The branched double cover of B branched along the two arcs is defined as the only branched cover such that its branch set is $a \cup b$, and the associate cover is the double cover given by the augmentation map (the map $\varepsilon : \pi_1(B \setminus (a \cup b)) \to \mathbb{Z}_2$ that sends both the meridians m_1 and m_2 that generate $\pi_1(B \setminus (a \cup b))$ into $1 \in \mathbb{Z}_2$).

This branched double cover can be constructed as follows. Let S be a surface in B whose boundary is made up of $a \cup b$ and two arcs on ∂B, as shown in Figure 1.2.

From now on, the ball B will be thought of as $B^2 \times I$, and the two arcs will be $\{x\} \times I$ and $\{y\} \times I$. Hence every (horizontal) section $B^2 \times \{t\}$ of B appears as in Figure 1.2.

A generator of $\pi_1(B \setminus S) \cong \mathbb{Z}$ is a loop around S and it is homotopic to $(m_1^{\pm 1} * m_2^{\pm 1})^{\pm 1}$, which is an element belonging to $\ker \varepsilon$. This implies that $\pi_1(B \setminus S) \subseteq \ker \varepsilon$. Thus, the restriction of the branched double cover to $B \setminus S$ (which is exactly the same as the restriction of the associate cover) is a trivial cover, since every loop in $B \setminus S$ lifts up to a closed path.

Hence, the restriction of the branched double cover to $B \setminus S$ is made up of two disjoint copies of $B \setminus S$. The branched double cover can now be obtained by gluing this two copies as shown in Figure 1.3.

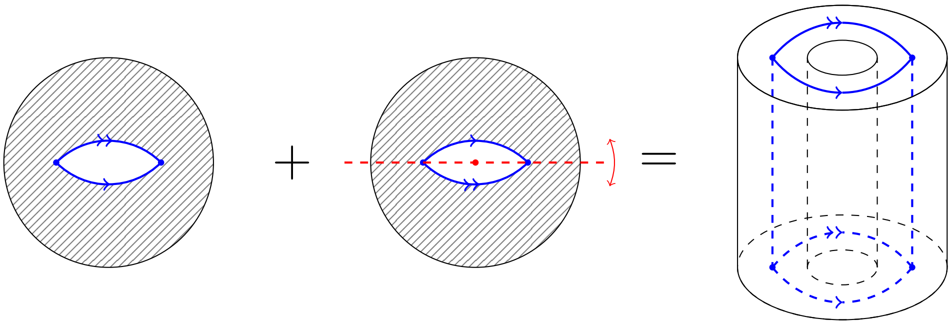


Figure 1.3: The branched double cover of the ball B branched along two arcs can be obtained by gluing two copies of $B \setminus S$. The picture shows the two copies seen from the top (or, equivalently, it shows the horizontal sections). A way to see the gluing is the following: imagine to do a circle inversion on the section on the right, and then a reflection along an axis belonging to the plane of the section; now the two sections can be glued simply by drawing the one on the right inside the hole of the one on the left. It is thus clear from the picture that the branched double cover of a ball branched along two untangled arcs is a solid torus.

Another way to see that the branched double cover of a ball branched along two untangled arcs is a solid torus is shown in Figure 1.4.

1.5 Triads

Triads are defined for instance in [OS04c], in which also the relevant properties are stated. In this section the concept of triad will be dealt in a detailed way.

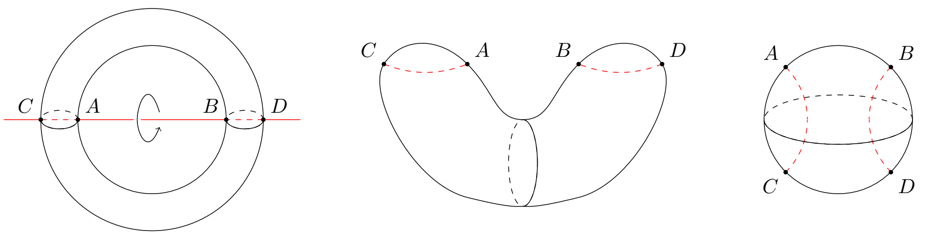


Figure 1.4: The quotient of a solid torus by the rotation of an angle π on the axis shown above is a ball. This shows that the branched double cover of a ball branched along two untangled arcs is a solid torus.

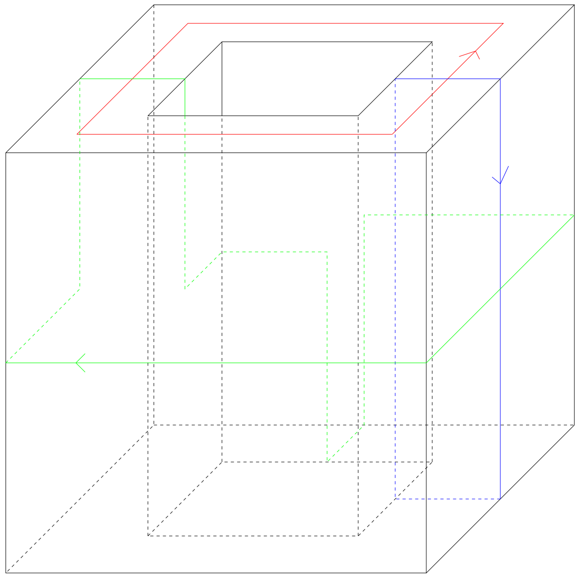


Figure 1.5: The picture shows $N(K) \subseteq Y$. The blue curve is $\gamma = \mu$, the red one is $\gamma_0 = \lambda$ and the green one is $\gamma_1 = -\lambda - \mu$.

1.5.1 The definition of triad

Definition 1.60. Let Y be a closed 3-manifold, and let K be a framed oriented knot in Y, with longitude λ and meridian μ. Let Y_0 be the 3-manifold obtained by λ-surgery on Y, and Y_1 the one obtained by $(\lambda + \mu)$-surgery on Y. Y can be seen as the μ-surgery on Y itself.

The triple (Y, Y_0, Y_1) is called a **triad** of 3-manifolds.

Remark. If the orientation of K is changed, the longitude λ and the meridian μ change orientation, but, as unoriented curves in Y, λ, μ and $\lambda + \mu$ do not change. Thus, the λ-surgery and the $(\lambda+\mu)$-surgery (as well as the manifolds Y_0 and Y_1) are well defined also if only an unoriented knot K together with an unoriented longitude λ is given.

If (Y, Y_0, Y_1) is a triad, the manifolds Y, Y_0 and Y_1 are obtained from $Y \setminus N(K)$ by Dehn filling. The meridian of the attaching solid torus is identified with, respectively, λ, μ and $\lambda + \mu$, as shown in Figure 1.5. On $\partial N(K)$, these three curves satisfy the following relations

$$\#(\mu \cap \lambda) = 1, \ \#(\lambda \cap (\lambda + \mu)) = -1, \ \#((\lambda + \mu) \cap \mu) = -1.$$

If $\gamma = \mu$, $\gamma_0 = \lambda$ and $\gamma_1 = -\lambda - \mu$, on $\partial N(K)$ the following relations hold

$$\#(\gamma \cap \gamma_0) = \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma) = 1,$$

which on $\partial(Y \setminus N(K))$ become

$$\#(\gamma \cap \gamma_0) = \#(\gamma_0 \cap \gamma_1) = \#(\gamma_1 \cap \gamma) = -1 \qquad (1.8)$$

since the orientation of the torus which bounds $Y \setminus N(K)$ is the opposite of the one that $N(K)$ induces on $\partial N(K)$.

Conversely, if M is a 3-manifold whose boundary is a torus, and if γ, γ_0 and γ_1 are three curves on ∂M satisfying (1.8), there is a solid torus that can be attached to M so that its meridian μ is γ and its longitude λ is γ_0. This implies also that $-\lambda - \mu$ is γ_1. Thus, if Y, Y_0 and Y_1 are the manifolds obtained by gluing a solid torus to M so that its meridian is respectively γ, γ_0 and γ_1, (Y, Y_0, Y_1) is a triad.

The previous reasoning can be summarized in the following proposition.

Proposition 1.61. *(Y, Y_0, Y_1) is a triad if and only if Y, Y_0 and Y_1 are obtained by Dehn filling on a manifold M, whose boundary is a torus, along curves γ, γ_0 and γ_1 so that Equation (1.8) holds.*

Corollary 1.62 (Invariance of a triad under cyclic permutation)**.** *(Y, Y_0, Y_1) is a triad if and only if (Y_0, Y_1, Y) is a triad.*

Now we will prove a result which relates the homology groups H_1 of three manifolds that fit into a triad.

Proposition 1.63. *Let (Y, Y_0, Y_1) be a triad. Then, up to a cyclic reordering,*

$$|\,\mathrm{H}_1(Y;\mathbb{Z})| = |\,\mathrm{H}_1(Y_0;\mathbb{Z})| + |\,\mathrm{H}_1(Y_1;\mathbb{Z})|,$$

where $|\,\mathrm{H}_1(\,\cdot\,;\mathbb{Z})|$ is defined in Definition 1.19.

Proof. By Theorem 1.11 Y_0 is the boundary of a 2-handlebody with only one 0-handle and no 1-handles X_0. Y_1 is obtained from Y_0 by Dehn surgery on a certain knot K with framing λ, so Y_1 is the boundary of the manifold obtained attaching to X_0 a 2-handle with attaching sphere K and framing λ. Analogously, Y is the boundary of the manifold obtained attaching to X_0 a 2-handle with attaching sphere K and framing $\lambda + \mu$, where μ is a meridian of K. Since we can suppose that the 2-handles are attached all together to B^4, λ is a framing in S^3, and so it is represented by a number $l \in \mathbb{Z}$. $\lambda + \mu$ is represented by $l + 1$.

By Proposition 1.27 the following matrices are the matrices of the intersection forms of respectively Y_0, Y_1 and Y:

$$Q_{X_0},\ \ Q_{X_1} = \begin{pmatrix} l & d^T \\ d & Q_{X_0} \end{pmatrix},\ \ Q_X = \begin{pmatrix} l+1 & d^T \\ d & Q_{X_0} \end{pmatrix}.$$

It is clear (by the multilinearity of the determinant) that

$$\det(Q_X) = \det(Q_{X_0}) + \det(Q_{X_1}). \tag{1.9}$$

Since by Lemma 1.18 Q_X (resp. Q_{X_0}, Q_{X_1}) is a presentation matrix of $\mathrm{H}_1(Y;\mathbb{Z})$ (resp. $\mathrm{H}_1(Y_0;\mathbb{Z})$, $\mathrm{H}_1(Y_1;\mathbb{Z})$), it follows that

$$\begin{aligned} |\det Q_X| &= |\,\mathrm{H}_1(Y;\mathbb{Z})|, \\ |\det Q_{X_0}| &= |\,\mathrm{H}_1(Y_0;\mathbb{Z})|, \\ |\det Q_{X_1}| &= |\,\mathrm{H}_1(Y_1;\mathbb{Z})|. \end{aligned}$$

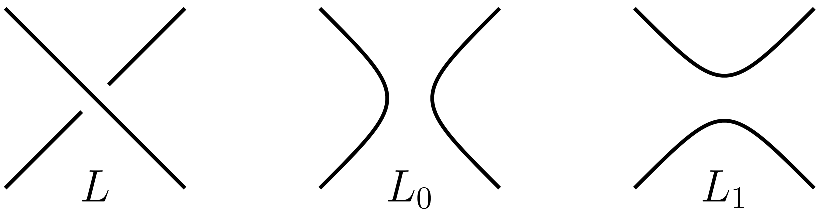


Figure 1.6: Definition of L_0 and L_1.

Note that the previous relations hold also if $\mathrm{H}_1(Y;\mathbb{Z})$ (resp. $\mathrm{H}_1(Y_0;\mathbb{Z})$, $\mathrm{H}_1(Y_1;\mathbb{Z})$) is not a finite set. Using the previous relations, Equation (1.9) becomes

$$\pm|H_1(Y;\mathbb{Z})| \pm |H_1(Y_0;\mathbb{Z})| \pm |H_1(Y_1;\mathbb{Z})| = 0.$$

Since the sum is 0, the three signs cannot be the same (or one of the summands must be 0). Thus, for some cyclic reordering of the triad (Y, Y_0, Y_1), we have that

$$|\,\mathrm{H}_1(Y;\mathbb{Z})| = |\,\mathrm{H}_1(Y_0;\mathbb{Z})| + |\,\mathrm{H}_1(Y_1;\mathbb{Z})|.$$

□

We have incidentally proved also the following result.

Corollary 1.64. *Let (Y_0, Y_1, Y) be a triad, and let X_0 be a 2-handlebody with only one 0-handle and no 1-handles such that $\partial X_0 = Y_0$. Let $K \subseteq S^3 = \partial B^4 \subseteq X_0$ be a knot with a certain framing $l \in \mathbb{Z}$. Let X_1 and X be the 2-handlebodies obtained from X_0 by attaching a 2-handle, so that the boundaries of the disks parallel to the core of the 2-handle are respectively the longitudes of K defined by the numbers l and $l+1$. Then $\partial X_1 = Y_1$ and $\partial X = Y$, and, if Q_{X_0} is a matrix representing the intersection form of X_0, the matrices representing the intersection forms of X_1 and X can be chosen as follows:*

$$Q_{X_1} = \begin{pmatrix} l & d^T \\ d & Q_{X_0} \end{pmatrix}, \quad Q_X = \begin{pmatrix} l+1 & d^T \\ d & Q_{X_0} \end{pmatrix}.$$

Moreover, the following relation holds:

$$\det(Q_X) = \det(Q_{X_0}) + \det(Q_{X_1}).$$

1.5.2 The skein triad

Let L be a link in S^3, with a fixed projection (a diagram) and a fixed crossing. Call L_0 and L_1 the two desingularizations of the diagram at the fixed crossing as it is shown in Figure 1.6.

Let

$$\pi : \Sigma(L) \to S^3$$

$$\pi_0 : \Sigma(L_0) \to S^3$$

$$\pi_1 : \Sigma(L_1) \to S^3$$

be the three branched double covers of L, L_0 and L_1.

Proposition 1.65. $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$ *is a triad, called the* ***skein triad****.*

Proof. Firstly note that, if B is a closed ball which is a 'neighbourhood' of the crossing, L_1 or L can be obtained from L_0 by removing B and attaching B again, but using an attaching diffeomorphism different from the identity map (since in all of the cases the link intersects the ball in two untangled arcs). As a result the three branched double covers over B are diffeomorphic:

$$\pi^{-1}(B) \cong \pi_0^{-1}(B) \cong \pi_1^{-1}(B).$$

The same holds also for the three branched double covers over $S^3 \setminus \mathring{B}$:

$$\pi^{-1}(S^3 \setminus \mathring{B}) \cong \pi_0^{-1}(S^3 \setminus \mathring{B}) \cong \pi_1^{-1}(S^3 \setminus \mathring{B}).$$

Thus the only difference between $\Sigma(L)$, $\Sigma(L_0)$ and $\Sigma(L_1)$ lies in the way the two pieces $\pi_0^{-1}(B)$ and $\pi_0^{-1}(S^3 \setminus \mathring{B})$ are attached together.

Example 1.59 shows that $\pi_0^{-1}(B)$ is a solid torus (that is called the L_0-solid torus). The construction in Figure 1.3 shows that the core of the L_0-solid torus (resp. L_1-solid torus, L-solid torus) is the branched double cover of an arc (the dashed arc in Figure 1.7) with endpoints on the two arcs of L_0 (resp. L_1, L). Now focus on the arc in ∂B obtained by pushing one of the arcs of L_0 (resp. L_1, L) on the boundary of B without intersecting the other arc: this arc is the blue arc (resp. the red one and the green one) shown in Figure 1.7. Again, the construction in Figure 1.3 shows that the branched double covers of the blue arc, the red one and the green one in the L_0-solid torus appear exactly as the blue loop, the red loop and the green loop in Figure 1.5.

If we call γ, γ_0 and γ_1 these three loops, we have that Equation (1.8) holds (provided that we have chosen the right orientations for γ, γ_0 and γ_1), and, thus, $(\Sigma(L_0), \Sigma(L_1), \Sigma(L))$ is a triad. □

1.6 Homology theories in low-dimensional topology

In this work some homology theories for 3-manifolds or for knots in S^3 are used. Since developing this theory would take too much space, the results that will be used are stated in this section.

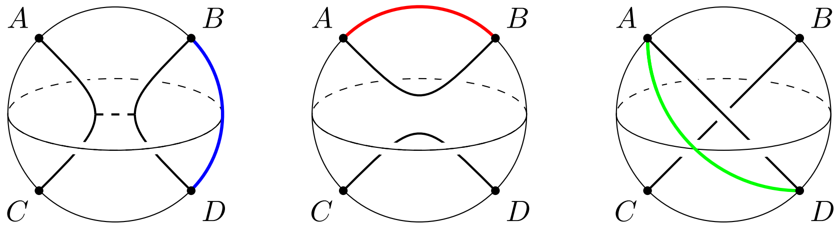


Figure 1.7: The ball B and the two arcs in the case of L_0, L_1 and L. The branched double cover of the dashed arc in the first picture is the core of the torus $\pi_0^{-1}(B)$. The branched double cover of the blue arc (resp. the red one and the green one) is a meridian of the torus $\pi_0^{-1}(B)$ (resp. $\pi_1^{-1}(B)$ and $\pi^{-1}(B)$).

1.6.1 Heegaard Floer homology

This subsection constitutes a brief introduction to Heegaard Floer homology. All the results will be stated without proof. For a more detailed discussion on this topic and for the missing proofs, see [OS04c].

Let Y be a closed 3-manifold. Then, Ozsváth and Szabó (cf. [OS04c]) associated to Y a $\mathbb{Z}_2$-vector space $\widehat{\mathrm{HF}}(Y)$, called the **Heegaard Floer homology** of Y.

The Heegaard Floer homology of Y has a $\mathbb{Z}_2$-grading, i.e.

$$\widehat{\mathrm{HF}}(Y) = \widehat{\mathrm{HF}}_0(Y) \oplus \widehat{\mathrm{HF}}_1(Y),$$

where both $\widehat{\mathrm{HF}}_0(Y)$ and $\widehat{\mathrm{HF}}_1(Y)$ are $\mathbb{Z}_2$-vector spaces. Moreover,

$$\widehat{\mathrm{HF}}(S^3) = \mathbb{Z}_2. \tag{1.10}$$

Theorem 1.66 (Exact triangle, [OS04c, Theorem 1.12])**.** *Let Y, Y_0 and Y_1 be three 3-manifolds that fit into a triad. Then there is a long exact sequence which relates their Heegaard Floer homologies (thought of as vector spaces over $\mathbb{Z}_2$):*

$$\dots \longrightarrow \widehat{\mathrm{HF}}(Y) \longrightarrow \widehat{\mathrm{HF}}(Y_0) \longrightarrow \widehat{\mathrm{HF}}(Y_1) \longrightarrow \dots$$

All the above maps respect the relative $\mathbb{Z}_2$ gradings, in the sense that each map carries homogeneous elements to homogeneous elements.

Definition 1.67. The **dimension** of the Heegaard Floer homology of a closed 3-manifold Y is its dimension as $\mathbb{Z}_2$-vector space. Hence

$$\dim(\widehat{\mathrm{HF}}(Y)) = \dim(\widehat{\mathrm{HF}}_0(Y)) + \dim(\widehat{\mathrm{HF}}_1(Y)).$$

The **Euler caracteristic** of the Heegaard Floer homology of a closed 3-manifold Y is the number

$$\chi(\widehat{\mathrm{HF}}(Y)) = \dim(\widehat{\mathrm{HF}}_0(Y)) - \dim(\widehat{\mathrm{HF}}_1(Y)).$$

Lemma 1.68 ([OS04c, Lemma 1.6]). *Let Y be a 3-manifold. Then*

$$\chi(\widehat{\mathrm{HF}}(Y)) = \pm|\,\mathrm{H}_1(Y;\mathbb{Z})|.$$

1.6.2 Homology theories for links

Some homology theories for links will be used in the following chapters, and they are introduced here. A brief introduction to these homology theories may be found also in [MO08].

Let L be a link in S^3. Its **Khovanov homology** is a bigraded $\mathbb{Z}$-module:

$$\mathrm{Kh}(L) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Kh}^{i,j}(L),$$

which was firstly defined by Khovanov in [Kho99]. An important property of Khovanov homology is that it is possible to recover the Jones polynomial of the link from it, so the following theorem holds.

Theorem 1.69. *Let L and L' be two links such that $\mathrm{Kh}(L) \cong \mathrm{Kh}(L')$. Then the Jones polynomials of L and L' are the same:*

$$V_L(t) = V_{L'}(t).$$

Another homology group that may be associated to a link L is **odd-Khovanov homology**, defined in [ORS07], which is a bigraded $\mathbb{Z}$-module:

$$\mathrm{Kh}^{\mathrm{odd}}(L) = \bigoplus_{i,j\in\mathbb{Z}} \mathrm{Kh}^{\mathrm{odd}^{i,j}}(L).$$

For both Khovanov and odd-Khovanov homology there exist 'reduced' versions Kh_r and $\mathrm{Kh}_r^{\mathrm{odd}}$. They are defined for example in [Ras05].

Finally, the **knot Floer homology** of link L in S^3 is a bigraded $\mathbb{Z}_2$-vector space:

$$\widehat{\mathrm{HFK}}(L) = \bigoplus_{i,j\in\mathbb{Z}} \widehat{\mathrm{HFK}}^{i,j}(L),$$

defined independently by Oszváth and Szabó in [OS04a] and by Rasmussen in [Ras03].

For each one of the three theories above, the δ-grading is defined as

$$\delta = j - i.$$

Definition 1.70. A link $L \subseteq S^3$ is **thin** if each one of its reduced Khovanov, reduced odd-Khovanov and knot Floer homology modules are free and supported in a single δ-grading.

Thin knots will be of crucial importance in the next chapters.

Chapter 2

Quasi-alternating links

In this chapter the concept of quasi-alternating link is introduced and discussed. It was shown in [MO08] that quasi-alternating links are thin. However, the converse does not hold: Greene proved in [Gre10] that the knot 11^n_{50} is not quasi-alternating, yet thin, and in [GW11] Greene and Watson furnished an infinite family of non-quasi-alternating thin knots.

The ultimate aim of this work is to exhibit other infinite families of non-quasi-alternating thin knots, using the techniques of [GW11]. To do so, in this chapter a necessary condition for being quasi-alternating is proved. In the following chapters we will construct families of thin knots that do not satisfy this condition and, thus, are not quasi-alternating.

2.1 The definition of quasi-alternating links

Definition 2.1. The set of **quasi-alternating links**, which is called $\mathcal{QA}$, is the smallest subset of the set of all links $\mathcal{L}$ that satisfies the following conditions:

1. the trivial knot $0_1 \in \mathcal{QA}$;
2. if the two desingularizations of a link L at a fixed crossing L_0 and L_1 (as in Figure 1.6) belong to $\mathcal{QA}$ and $\det(L) = \det(L_0) + \det(L_1)$, then $L \in \mathcal{QA}$.

Remark. Recall that the smallest subset S of a set X satisfying a certain property $\mathcal{P}$ is defined as the intersection of all subsets of X satisfying $\mathcal{P}$:

$$S = \bigcap_{T \subseteq X,\, T \text{ satisfies } \mathcal{P}} T.$$

Remark. The given definition of $\mathcal{QA}$ is an abstract definition. However, the set $\mathcal{QA}$ can also be constructed inductively. Let E_n be the subsets of the set of all links $\mathcal{L}$ defined inductively as follows:

$$E_1 = \{0_1 = \text{the trivial knot}\};$$

$$E_n = \left\{ L \in \mathcal{L} \;\middle|\; \begin{array}{c} \det L = n, \exists \text{ a crossing of some diagram of } L, \\ \exists i, j \geq 1\colon\ i + j = n,\ L_0 \in E_i,\ L_1 \in E_j \end{array} \right\}.$$

By induction on n, $E_n \subseteq \mathcal{QA}\ \forall\, n \geq 1$, so $\bigcup E_n \subseteq \mathcal{QA}$. Conversely, $\bigcup E_n$ is a subset of $\mathcal{L}$ satisfying the two conditions of Definition 2.1, and so $\mathcal{QA} \subseteq \bigcup E_n$. Hence it clearly follows that

$$\mathcal{QA} = \bigcup_{n \geq 1} E_n.$$

Corollary 2.2. *Let $L \subseteq S^3$ be a link.*

1. *If $L \in \mathcal{QA}$, then $\det L \geq 1$.*
2. *If $L \in \mathcal{QA}$ and $\det L = 1$, then L is the trivial knot.*

Proof. For the first statement, if $L \in \mathcal{QA}$, then $L \in E_n$ for some $n \geq 1$, and so $\det L = n \geq 1$. For the second statement, if $L \in \mathcal{QA}$ and $\det L = 1$, then $L \in E_1 = \{0_1\}$. Thus L is the trivial knot. □

Links which admit a connected alternating projection are quasi-alternating due to a result by Ozsváth and Szabó (cf. [OS05, Lemma 3.2]), so quasi-alternating links are a generalization of alternating knots.

Another interesting property of quasi-alternating links, which is proved in [MO08], is stated in the following theorem.

Theorem 2.3. *Quasi-alternating links are thin.*

2.2 L-spaces

The necessary condition for $\mathcal{QA}$-ness that will be proved in this chapter is a consequence of some properties of the branched double cover of S^3 along quasi-alternating links, which are stated in Theorem 2.7. One of these properties is that the branched double cover of a quasi-alternating link is always an L-space. This is the reason why this section is devoted to L-spaces.

Definition 2.4. A closed 3-manifold Y is an **L-space** if it is a rational homology sphere (i.e. $\mathrm{H}_*(Y;\mathbb{Q}) \cong \mathrm{H}_*(S^3;\mathbb{Q})$) and $|\mathrm{H}_1(Y;\mathbb{Z})| = \dim(\widehat{\mathrm{HF}}(Y))$.

Lemma 2.5. *Let (Y, Y_0, Y_1) be a triad, ordered so that*

$$|\mathrm{H}_1(Y;\mathbb{Z})| = |\mathrm{H}_1(Y_0;\mathbb{Z})| + |\mathrm{H}_1(Y_1;\mathbb{Z})|. \tag{2.1}$$

If Y_0 and Y_1 are L-spaces, so is Y.

Remark. Recall that by Proposition 1.63 there must be a cyclic reordering so that Equation (2.1) holds.

Proof of Lemma 2.5. First note that, since Y_0 and Y_1 are rational homology spheres, $|\,\mathrm{H}_1(Y_0;\mathbb{Z})|$ and $|\,\mathrm{H}_1(Y_1;\mathbb{Z})|$ are finite and $\neq 0$, then, by Equation (2.1), so is $|\,\mathrm{H}_1(Y;\mathbb{Z})|$. This proves that $\mathrm{H}_1(Y;\mathbb{Q}) = 0$ and, by Poincaré duality, $\mathrm{H}_2(Y;\mathbb{Q}) = 0$. Connectedness and orientability of Y follow from connectedness and orientability of Y_0 or Y_1, so $\mathrm{H}_0(Y;\mathbb{Q}) = \mathbb{Q}$ and $\mathrm{H}_3(Y;\mathbb{Q}) = \mathbb{Q}$. Thus, Y is a rational homology sphere.

Now we only have to prove that $|\,\mathrm{H}_1(Y;\mathbb{Z})| = \dim(\widehat{\mathrm{HF}}(Y))$. We will prove the two inequalities.

By Lemma 1.68

$$\begin{aligned}
|\,\mathrm{H}_1(Y;\mathbb{Z})| &= \pm\chi(\widehat{\mathrm{HF}}(Y)) \\
&= \pm\left(\dim(\widehat{\mathrm{HF}}_0(Y)) - \dim(\widehat{\mathrm{HF}}_1(Y))\right) \\
&\leq \dim(\widehat{\mathrm{HF}}_0(Y)) + \dim(\widehat{\mathrm{HF}}_1(Y)) \\
&\leq \dim(\widehat{\mathrm{HF}}(Y))
\end{aligned}$$

so we have the first inequality

$$|\,\mathrm{H}_1(Y;\mathbb{Z})| \leq \dim(\widehat{\mathrm{HF}}(Y)). \tag{2.2}$$

For the second inequality, by the exact triangle in Heegaard Floer homology (Theorem 1.66) it follows that

$$\dim(\widehat{\mathrm{HF}}(Y)) \leq \dim(\widehat{\mathrm{HF}}(Y_0)) + \dim(\widehat{\mathrm{HF}}(Y_1)).$$

Since Y_0 and Y_1 are L-spaces, the last equation is equivalent to

$$\dim(\widehat{\mathrm{HF}}(Y)) \leq |\,\mathrm{H}_1(Y_0;\mathbb{Z})| + |\,\mathrm{H}_1(Y_1;\mathbb{Z})| = |\,\mathrm{H}_1(Y;\mathbb{Z})|, \tag{2.3}$$

where the last equality is Equation (2.1).

Equations (2.2) and (2.3) prove the Lemma. □

2.3 A result on the branched double cover of a quasi-alternating link

As already anticipated, if a link L is quasi-alternating, there is a strong condition on its branched double cover $\Sigma(L)$, which is stated in Theorem 2.7. For the proof of the Theorem we will need the following Lemma.

Lemma 2.6. *Let $A \in M_{n-1}(\mathbb{R})$ a symmetric matrix $(n-1)\times(n-1)$. Let $B \in M_n(\mathbb{R})$ a matrix such that*

$$B = \begin{pmatrix} c & d^T \\ d & A \end{pmatrix}$$

for some $c \in \mathbb{R}$ and $d \in \mathbb{R}^{n-1}$. Suppose that $\det A \neq 0$ *and* $\det B \neq 0$.

1. *If the signs of* $\det A$ *and* $\det B$ *are the same, then the signature of* B *is*
$$(b_+(B), b_-(B)) = (b_+(A) + 1, b_-(A)).$$

2. *If the signs of* $\det A$ *and* $\det B$ *are different, then the signature of* B *is*
$$(b_+(B), b_-(B)) = (b_+(A), b_-(A) + 1).$$

Proof. First note that, since b_+ (resp. b_-) is the dimension of a maximal subspace where the form defined by the matrix is positive (resp. negative) defined, we have that

$$b_+(B) \geq b_+(A), \;\; b_-(B) \geq b_-(A).$$

As $\mathrm{rk}(B) = \mathrm{rk}(A) + 1$ and neither A nor B have an eigenvector with eigenvalue 0 (since $\det A \neq 0$ and $\det B \neq 0$), it follows that the two possible cases are:

$$(b_+(B), b_-(B)) = (b_+(A) + 1, b_-(A));$$
$$(b_+(B), b_-(B)) = (b_+(A), b_-(A) + 1).$$

As the determinant is the product of the eigenvalues (with multiplicity), if the signs of $\det A$ and $\det B$ are the same, this means that $b_-(B) = b_-(A)$ and so we are in the first case; if instead the signs of $\det A$ and $\det B$ are different, this means that $b_-(B) = b_-(A) + 1$ and so we are in the second case. □

Theorem 2.7. *If* $L \in \mathcal{QA}$, *then* $\Sigma(L)$ *is an* L*-space and it is the boundary of a negative definite* 2*-handlebody* X *with only one* 0*-handle and no* 1*-handles, and such that* $b_2(X) < \det(L)$.

Remark. $b_2(X)$ represents the second Betti number of X, i.e. the rank of the $\mathbb{Z}$-module $\mathrm{H}_2(X; \mathbb{Z})$.

Remark. Theorem 2.7 is a slightly stronger version of Theorem 4 in [GW11]. Greene and Watson proved indeed that $\Sigma(L)$ is the boundary of a negative 4-manifold X such that $\mathrm{H}_1(X; \mathbb{Z}) = 0$, whereas we will prove that as X we can choose a 2-handlebody with only one 0-handle and no 1-handles.

Proof of Theorem 2.7. We argue by induction on $\det L$. If $\det L = 1$, by Corollary 2.2 L is the trivial knot. Thus, $\Sigma(L)$ is S^3 (and so it is an L-space by Equation (1.10)), and $X = B^4$ clearly satisfies the required properties.

For the inductive step, let L_0 and L_1 be the two resolutions of L given by Definition 2.1. Since $\det L = \det L_0 + \det L_1$, and $\det L_0$ and $\det L_1$ are strictly positive by Corollary 2.2, it is clear that

$$\det L_0 < \det L, \;\; \det L_1 < \det L.$$

Thus the inductive hypothesis can be applied to L_0 and L_1.

By Lemma 1.58 the relation $\det L = \det L_0 + \det L_1$ becomes

$$|\,\mathrm{H}_1(\Sigma(L);\mathbb{Z})| = |\,\mathrm{H}_1(\Sigma(L_0);\mathbb{Z})| + |\,\mathrm{H}_1(\Sigma(L_1);\mathbb{Z})|. \tag{2.4}$$

By Proposition 1.65 $(\Sigma(L), \Sigma(L_0), \Sigma(L_1))$ is a triad. Moreover, since $\Sigma(L_0)$ and $\Sigma(L_1)$ are L-spaces by inductive hypothesis, by Lemma 2.5 so is $\Sigma(L)$.

Now we have to show that $\Sigma(L)$ bounds a negative definite 2-handlebody with only one 0-handle and no 1-handles. By inductive hypothesis $\Sigma(L_1)$ is the boundary of a negative definite 2-handlebody X_1 with only one 0-handle and no 1-handles. Since $(\Sigma(L_1), \Sigma(L), \Sigma(L_0))$ is a triad, by Corollary 1.64 applied to the triad $(Y_0, Y_1, Y) = (\Sigma(L_1), \Sigma(L), \Sigma(L_0))$ (be careful on the order of the triad!), we can construct two 2-handlebodies X and X_0 by attaching a 2-handle to X_1, so that $\partial X = \Sigma(L)$, $\partial X_0 = \Sigma(L_0)$ and, if Q_{X_0}, Q_{X_1} and Q_X are matrices representing the intersection forms of X_0, X_1 and X, the following relations hold:

$$Q_X = \begin{pmatrix} l & d^T \\ d & Q_{X_1} \end{pmatrix}, \quad Q_{X_0} = \begin{pmatrix} l+1 & d^T \\ d & Q_{X_1} \end{pmatrix},$$

$$\det(Q_{X_0}) = \det(Q_{X_1}) + \det(Q_X). \tag{2.5}$$

Moreover, as X and X_0 are obtained from X_1 by attaching a 2-handle, the following relations on the Betti numbers hold:

$$b_2(X) = b_2(X_0) = b_2(X_1) + 1. \tag{2.6}$$

By Equation (2.4) and Corollary 1.20, it follows that

$$|\det Q_{X_0}| = |\det Q_X| - |\det Q_{X_1}|. \tag{2.7}$$

Combining Equations (2.5) and (2.7) we have that

$$|\det Q_X + \det Q_{X_1}| = |\det Q_X| - |\det Q_{X_1}|.$$

Since $\det Q_{X_1} \neq 0$ (as X_1 is negative definite), the above equation implies that also $\det Q_X \neq 0$ and the signs of $\det Q_X$ and $\det Q_{X_1}$ are opposite.

By Lemma 2.6 the signature of Q_X is

$$\begin{aligned}(b_+(Q_X), b_-(Q_X)) &= (b_+(Q_{X_1}), b_-(Q_{X_1}) + 1) \\ &= (0, b_2(X_1) + 1),\end{aligned}$$

so Q_X is negative definite.

Finally, the fact that $b_2(X) < \det L$ can be proved using Equation (2.6) and the inductive hypothesis on X_1:

$$b_2(X) = b_2(X_1) + 1 < \det L_1 + 1 \leq \det L_1 + \det L_0 = \det L.$$

□

2.4 An obstruction to $\mathcal{QA}$-ness

In this section an obstruction to $\mathcal{QA}$-ness is stated and proved. The proof of this obstruction depends on some important results: specifically, on Theorem 2.7, which was proved in Section 2.3, and on the following two lemmas.

Lemma 2.8 (Eisenstein-Hermite, [MH73, Lemma 1.6]). *For each constant $M > 0$, there are finitely many definite, integral lattices (up to isomorphism) with rank and discriminant less than M.*

Remark. Recall that the **discriminant** of an integral lattice is the determinant of a matrix representing the bilinear form. It is well defined because the determinant of any matrix representing a change of basis on a free $\mathbb{Z}$-module is ± 1.

Before stating the next lemma, an introduction is required. Let X be an oriented connected 4-manifold whose boundary $Y = \partial X$ is a rational homology sphere. Then each class of $\mathrm{H}_1(Y;\mathbb{Z})$ is a torsion class, so, by the long exact sequence of the pair

$$\ldots \longrightarrow \mathrm{H}_2(X;\mathbb{Z}) \longrightarrow \mathrm{H}_2(X,Y;\mathbb{Z}) \longrightarrow \mathrm{H}_1(Y;\mathbb{Z}) \longrightarrow \ldots,$$

for each class A in $\mathrm{H}_2(X,Y;\mathbb{Z})$ there exists an integer $k > 0$ such that $k \cdot A$ comes from a class in $\mathrm{H}_2(X;\mathbb{Z})$. Up to Poincaré duality, this means that for each cohomology class $\xi \in \mathrm{H}^2(X;\mathbb{Z})$ there exists a multiple $k \cdot \xi$ that comes from $\mathrm{H}^2(X,Y;\mathbb{Z})$.

For each class $\eta \in \mathrm{H}^2(X,Y;\mathbb{Z})$, the cup product $\eta^2 = \eta \smile \eta$ can be thought of as an integer since $\mathrm{H}^4(X,Y;\mathbb{Z}) \cong \mathrm{H}_0(X;\mathbb{Z}) \cong \mathbb{Z}$ in a canonical way. So also $k \cdot \xi \smile k \cdot \xi$ can be thought of as an integer. Hence we can define, for each $\xi \in \mathrm{H}^2(X;\mathbb{Z})$,

$$\xi^2 = \frac{1}{k^2}(k \cdot \xi \smile k \cdot \xi) \in \mathbb{Q}.$$

Ozsváth and Szabó defined in [OS03] an invariant of a 3-manifold Y endowed with a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{t}$, which is known as the **correction term** or the ***d*-invariant** $d(Y,\mathfrak{t}) \in \mathbb{Q}$.

Lemma 2.9 (Ozsváth-Szabó, [OS03, Theorem 9.6]). *Suppose that Y is a rational homology sphere which bounds a 4-dimensional oriented, connected manifold X with negative definite intersection form and $\mathrm{H}_1(X;\mathbb{Z}) = 0$. Then, every Spin$^{\mathbb{C}}$ structure $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(Y)$ extends to some $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ and for each such extension the following inequality holds:*

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y,\mathfrak{t}).$$

Remark. The first part of Lemma 2.9, i.e. the fact that every $\mathrm{Spin}^{\mathbb{C}}$ structure on Y can be extended to X, was already proved in Corollary 1.54 in the special case when X is a 2-handlebody with only one 0-handle and no 1-handles.

Now the obstruction to $\mathcal{QA}$-ness can be stated and proved.

Theorem 2.10 ([GW11, Proposition 3]). *For each integer $D \geq 1$, there exists a constant $C(D) \in \mathbb{Z}$ such that for every quasi-alternating link L with* $\det L = D$ *and* $\forall\, \mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L))$

$$C \leq d(\Sigma(L), \mathfrak{t}). \tag{2.8}$$

Proof. Let L be an arbitrary quasi-alternating link with $\det L = D$, and let X be a 2-handlebody that bounds $\Sigma(L)$ as in Theorem 2.7. Finally, let $\Lambda = (Z, f)$ denote the lattice $(\mathrm{H}_2(X;\mathbb{Z}), Q_X)$.

Recall that by Theorem 1.51 c_1 is a bijection between $\mathrm{Spin}^{\mathbb{C}}(X)$ and $\mathrm{Char}(\Lambda)$. Let $\mathrm{Char}(\Lambda, \mathfrak{t})$ denote the subset of $\mathrm{Char}(\Lambda)$ of the characteristic classes corresponding to $\mathrm{Spin}^{\mathbb{C}}$ structures on X that restrict to $\mathfrak{t}$ on $\Sigma(L)$ (the restriction map is the one defined in Theorem 1.47):

$$\mathrm{Char}(\Lambda, \mathfrak{t}) = \left\{ c_1(\mathfrak{s}) \middle| \, \mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X),\ \mathfrak{s}_{|_{\Sigma(L)}} = \mathfrak{t} \right\}.$$

The partition of $\mathrm{Char}(\Lambda)$ into the subsets $\mathrm{Char}(\Lambda, \mathfrak{t})$ actually does not depend on $\Sigma(L)$ and its $\mathrm{Spin}^{\mathbb{C}}$ structures, it depends only on the algebraic structure of the lattice Λ: the subsets $\mathrm{Char}(\Lambda, \mathfrak{t})$ are indeed the equivalence classes of the action of $2 \cdot \widehat{f}(Z)$ on $\mathrm{Char}(\Lambda)$ (the notation is the same as in Lemma 1.52). To prove this, recall that by Theorem 1.53 there is a bijection between $\mathrm{Spin}^{\mathbb{C}}(\Sigma(L))$ and $\mathrm{Char}(\Lambda)/(2 \cdot \widehat{f}(Z))$ that is equivariant for the action of $Z^*/\widehat{f}(Z)$. Let $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L))$ be a $\mathrm{Spin}^{\mathbb{C}}$ structure on $\Sigma(L)$, and let $[\chi]$ denote the corresponding class in $\mathrm{Char}(\Lambda)/(2 \cdot \widehat{f}(Z))$. Since the following diagram commutes (cf. Theorem 1.53),

$$\begin{tikzcd}
 & \mathrm{Spin}^{\mathbb{C}}(X) \arrow[dl, "\cdot\,|_{\Sigma(L)}"'] \arrow[dr, "{[c_1(\cdot)]}"] & \\
\mathrm{Spin}^{\mathbb{C}}(\Sigma(L)) \arrow[rr, "\sim"] & & \mathrm{Char}(\Lambda)/(2 \cdot \widehat{f}(Z))
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathfrak{s} \arrow[dl, mapsto, "\cdot\,|_{\Sigma(L)}"'] \arrow[dr, mapsto, "{[c_1(\cdot)]}"] & \\
\mathfrak{t} \arrow[rr, mapsto] & & {[c_1(\mathfrak{s})]}
\end{tikzcd}$$

a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ restricts to $\mathfrak{t}$ if and only if $[c_1(\mathfrak{s})] = [\chi]$. Hence, if $\mathfrak{s}$ and $\widetilde{\mathfrak{s}} \in \mathrm{Spin}^{\mathbb{C}}(X)$, $c_1(\mathfrak{s})$ and $c_1(\widetilde{\mathfrak{s}})$ belong to the same $\mathrm{Char}(\Lambda, \mathfrak{t})$ if and only if $[c_1(\mathfrak{s})] = [c_1(\widetilde{\mathfrak{s}})]$, so the subsets $\mathrm{Char}(\Lambda, \mathfrak{t})$ are exactly the classes of $\mathrm{Char}(\Lambda)$ modulo $(2 \cdot \widehat{f}(Z))$.

Moreover, there are exactly D equivalence classes $\mathrm{Char}(\Lambda, \mathfrak{t})$. They are indeed as many as the $\mathrm{Spin}^{\mathbb{C}}$ structures on $\Sigma(L)$, so, as $\mathrm{Spin}^{\mathbb{C}}(\Sigma(L))$ is an affine space over $\mathrm{H}^2(\Sigma(L);\mathbb{Z})$, they are $|\,\mathrm{H}^2(\Sigma(L);\mathbb{Z})|$. Then, by Poincaré duality and Lemma 1.58, it follows that they are D:

$$|\,\mathrm{H}^2(\Sigma(L);\mathbb{Z})| = |\,\mathrm{H}_1(\Sigma(L);\mathbb{Z})| = \det L = D.$$

Hence, if $\chi_1, \dots, \chi_D \in \mathrm{Char}(\Lambda)$ represent the classes of $\mathrm{Char}(\Lambda)$ modulo $(2 \cdot \widehat{f}(Z))$, the two partitions

$$\mathrm{Char}(\Lambda) = \coprod_{\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L))} \mathrm{Char}(\Lambda, \mathfrak{t}) = \coprod_{j=1}^{D} \Big(\chi_j + 2 \cdot \widehat{f}(Z)\Big)$$

are actually the same.

Let $M(\Lambda, \mathfrak{t})$ be the number defined by

$$M(\Lambda, \mathfrak{t}) = \sup \left\{ \frac{\chi^2 + \mathrm{rk}(\Lambda)}{4} \in \frac{1}{4}\mathbb{N} \,\middle|\, \chi \in \mathrm{Char}(\Lambda, \mathfrak{t}) \right\}.$$

Lemma 2.9 assures that

$$M(\Lambda, \mathfrak{t}) \leq d(\Sigma(L), \mathfrak{t}). \tag{2.9}$$

Moreover, let $m(\Lambda)$ be the minimum

$$m(\Lambda) = \min \Big\{ M(\Lambda, \mathfrak{t}) \,\Big|\, \mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L)) \Big\},$$

which exists because $|\,\mathrm{Spin}^{\mathbb{C}}(\Sigma(L))| = D$, so the minimum is taken over a finite set.

More generally, for an integral negative definite lattice $\Lambda' = (Z', f')$ we can define

$$M(\Lambda', [\overline{\chi}]) = \sup \left\{ \frac{(f')^*(\chi,\chi) + \mathrm{rk}(\Lambda')}{4} \,\middle|\, \chi \in \Big(\overline{\chi} + 2 \cdot \widehat{f'}(Z')\Big) \right\},$$

$$\overline{m}(\Lambda') = \inf \Big\{ M(\Lambda', [\overline{\chi}]) \,\Big|\, [\overline{\chi}] \in {}^{\mathrm{Char}(\Lambda')}\big/_{(2 \cdot \widehat{f'}(Z'))} \Big\}.$$

$(f')^*$ is here the form on the dual space Z'^* associated to f': as f' is non-degenerate (because it is definite), there exists a unique bilinear form $(f')^*$ on $(Z')^*$ such that $\forall\, v, w \in Z'$

$$(f')^*(\widehat{f'}(v), \widehat{f'}(w)) = f'(v,w). \tag{2.10}$$

Suppose now that the lattice $\Lambda' = (Z', f')$ is compatible with some L', i.e. there exist a quasi-alternating link L' and a negative definite 2-handlebody X' with only one 0-handle and no 1-handles such that the boundary of X' is $\Sigma(L')$, $b_2(X') < \det L'$ and $\Lambda' = (\mathrm{H}_2(X';\mathbb{Z}), Q_{X'})$.

Then each characteristic class is the Chern class of a $\mathrm{Spin}^{\mathbb{C}}$ structure on X' (cf. Theorem 1.51), and the classes $[\overline{\chi}] \in \mathrm{Char}(\Lambda')/(2 \cdot \widehat{f'}(Z'))$ are the subsets $\mathrm{Char}(\Lambda', \mathfrak{t})$. Moreover

$$\begin{aligned}
c_1(\mathfrak{s})^2 &= \frac{1}{k^2}\langle (k \cdot c_1(\mathfrak{s}))^2 \, , \big[X', \Sigma(L')\big]\rangle \\
&= \frac{1}{k^2}\langle \mathrm{PD}^{-1}(\mathrm{PD}(k \cdot c_1(\mathfrak{s}))) \smile \mathrm{PD}^{-1}(\mathrm{PD}(k \cdot c_1(\mathfrak{s}))), \big[X', \Sigma(L')\big]\rangle \\
&= \frac{1}{k^2} Q_{X'}(\mathrm{PD}(k \cdot c_1(\mathfrak{s})), \mathrm{PD}(k \cdot c_1(\mathfrak{s}))) \\
&= \frac{1}{k^2}(f')^*(\widehat{f'}(\mathrm{PD}(k \cdot c_1(\mathfrak{s}))), \widehat{f'}(\mathrm{PD}(k \cdot c_1(\mathfrak{s})))) \\
&= \frac{1}{k^2}(f')^*(k \cdot c_1(\mathfrak{s}), k \cdot c_1(\mathfrak{s})), \\
&= (f')^*(c_1(\mathfrak{s}), c_1(\mathfrak{s})),
\end{aligned}$$

where the fourth equality is Equation (2.10) and the fifth one is due to the commutativity of Diagram 1.2. Since $c_1(\mathfrak{s})^2 = (f')^*(c_1(\mathfrak{s}), c_1(\mathfrak{s}))$, the sets

$$\left\{ M(\Lambda', \mathfrak{t}) \,\middle|\, \mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L')) \right\}$$

and

$$\left\{ M(\Lambda', [\overline{\chi}]) \,\middle|\, [\overline{\chi}] \in \mathrm{Char}(\Lambda') \big/ (2 \cdot \widehat{f'}(Z')) \right\}$$

are in fact the same. Hence $m(\Lambda') = \overline{m}(\Lambda')$ and this number depends only on the algebraic structure of the lattice Λ' (and not on $\Sigma(L')$ or X').

Now consider the quantity

$$C(D) = \min \left\{ m(\Lambda') \,\middle|\, \begin{array}{c} \Lambda' \text{ integral, negative defined lattice} \\ \text{compatible with some } \Sigma(L') \\ \mathrm{rk}(\Lambda') < \mathrm{disc}(\Lambda') = D \end{array} \right\}.$$

The minimum exists because it is taken over a finite set by Lemma 2.8.

The lattice Λ belongs to the set over which the minimum $C(D)$ is taken: indeed by Theorem 2.7 Λ is negative definite and $\mathrm{rk}(\Lambda) < D$; the discriminant of Λ is D because, if Q is a matrix representing f, by Corollary 1.20 and Lemma 1.58

$$\mathrm{disc}(\Lambda) = \det Q = |\,\mathrm{H}_1(\Sigma(L))| = \det L = D.$$

As Λ belongs to the set over which the minimum $C(D)$ is taken, we have

$$C(D) \leq m(\Lambda) \leq M(\Lambda, \mathfrak{t}) \quad \forall\, \mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L)).$$

Combining the last equation with Equation (2.9), we have

$$C(D) \leq d(\Sigma(L), \mathfrak{t}) \quad \forall\, \mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(L)).$$

Since L was arbitrary, the theorem is proved. $\square$

Chapter 3

Turaev torsion

This chapter is focused on Turaev torsion and the concepts required to define it.

First, the concept of Euler structure is introduced. There are several definitions of Euler structure, and all of them are equivalent to each other. In the case of closed connected 3-manifolds, Euler structures are also equivalent to $\mathrm{Spin}^{\mathbb{C}}$ structures. All these properties of Euler structures will be discussed in Sections 3.1 and 3.2.

After discussing Euler structures, Turaev torsion is introduced. It is possible to associate to any Euler structure on a closed homologically oriented manifold a rational number, which is its Reidemeister-Franz torsion. Turaev torsion is an invariant that in some way gathers the information given by the torsions of all Euler structures on a certain manifold.

Finally, an explicit way to calculate the Turaev torsion starting from a presentation of the fundamental group of the manifold is explained.

In Chapter 4 some infinite families of non-quasi-alternating thin knots will be exhibited. Turaev torsion will allow us to distinguish the knots of these families from each other, and to prove that the knots are not quasi-alternating. Indeed, Theorem 2.10 provides a lower bound on the correction term $d(\Sigma(L), \mathfrak{t})$ if L is quasi-alternating, which results in a lower bound on each coefficient of the Turaev torsion of $\Sigma(L)$. We will exhibit infinite families of thin knots such that the coefficients of the Turaev torsion of their branched double covers are arbitrarily low, so an infinite subfamily of each of these families must be non-quasi-alternating.

3.1 Euler structures

3.1.1 Combinatorial Euler structures

Let A be a finite connected CW complex and let $E = \{e \mid e \text{ open cell of } A\}$. An **Euler chain** in A is a singular chain $\vartheta \in \mathrm{C}_1(A;\mathbb{Z})$ such that

$$\partial\vartheta = \sum_{e\in E} (-1)^{\dim e} x_e,$$

where x_e denotes the central point of the cell e.

Remark. An Euler chain exists if and only if the Euler characteristic of the manifold is 0 (that is if and only if it is possible to find a bijective correspondence between the even-dimensional cells and the odd-dimensional cells).

If ϑ and η are two Euler chains, then the difference $\vartheta - \eta$ is a 1-cycle in A. Consider the following equivalence relation:

$$\vartheta \sim \eta \Longleftrightarrow \vartheta - \eta \in \mathrm{B}_1(A;\mathbb{Z}).$$

Definition 3.1. Equivalence classes of Euler chains under the equivalence relation $\sim$ are called (**combinatorial**) **Euler structures** on A. The set of Euler structures is denoted by $\mathrm{Eul}(A)$.

Lemma 3.2. $\mathrm{Eul}(A)$ *is endowed with a natural free and transitive action of* $\mathrm{H}_1(A;\mathbb{Z})$.

Proof. $\mathrm{Z}_1(A;\mathbb{Z})$ acts on $\mathrm{Eul}(A)$ with the standard action

$$h\cdot[\vartheta] = [\vartheta + h]\,.$$

As by definition the boundaries act trivially on $\mathrm{Eul}(A)$, the action descends to the quotient action

$$\mathrm{H}_1(A;\mathbb{Z}) \times \mathrm{Eul}(A) \longrightarrow \mathrm{Eul}(A).$$

The quotient action is transitive because the difference of two Euler chains is always a 1-cycle, and it is free because two Euler chains are equivalent only if their difference is a boundary. $\square$

Remark. If $\chi(A) = 0$ (i.e. if there exists at least one Euler structure), then Lemma 3.2 implies that $\mathrm{Eul}(A)$ is an affine space on $\mathrm{H}_1(A;\mathbb{Z})$.

Subdivisions

One of the most common operation on CW complexes is the cellular subdivision. As we will see, there exists a natural bijective correspondence between the Euler structures on a CW complex and the Euler structures of a cellular subdivision of it.

Theorem 3.3 ([Tur90, §1.2])**.** *Let A be a finite CW complex, and let B be a cellular subdivision of A. Then there exists a canonical $H_1(A)$-equivariant bijection between* $\mathrm{Eul}(A)$ *and* $\mathrm{Eul}(B)$.

Proof. For each cell $b \in B$ there exists a unique cell $a(b)$ such that as topological spaces $b \subseteq a(b)$. Let γ_b be a 1-chain such that

$$\partial\gamma_b = x_b - x_a,$$

where x_a and x_b denote the centres of the cells a and b.

For each Euler chain $\vartheta \in \mathrm{Eul}(A)$, let $\sigma(\vartheta)$ be the 1-chain defined by

$$\sigma(\vartheta) = \vartheta + \sum_{b\in B}(-1)^{\dim b}\gamma_b.$$

As we will see, $\sigma(\vartheta)$ is an Euler chain in B. Indeed

$$\begin{aligned}\partial(\sigma(\vartheta)) &= \sum_{a\in A}(-1)^{\dim a}x_a + \sum_{b\in B}(-1)^{\dim b}x_b - \sum_{b\in B}(-1)^{\dim b}x_{a(b)}\\ &= \sum_{a\in A}\left[(-1)^{\dim a} - \sum_{\substack{b\in B\\ b\subseteq a}}(-1)^{\dim b}\right]x_a + \sum_{b\in B}(-1)^{\dim b}x_b. \end{aligned}\tag{3.1}$$

Consider the expression in square bracket. The two expressions

$$(-1)^{\dim a} \text{ and } \sum_{\substack{b\in B\\ b\subseteq a}}(-1)^{\dim b}$$

are the relative Euler characteristic $\chi(A^{\dim a}, A^{\dim a}\setminus a)$ (where $A^{\dim a}$ denotes the $(\dim a)$-skeleton of A), so their difference is zero. In other words, consider the quotient of $A^{\dim a}$ by the complementary of a. The two CW structures induced by A and B on the quotient furnish two expressions of the Euler characteristic of the quotient:

$$1 + (-1)^{\dim a} = \chi\left(A^{\dim a}/(A^{\dim a}\setminus a)\right) = 1 + \sum_{\substack{b\in B\\ b\subseteq a}}(-1)^{\dim b},$$

so the expression in the square bracket in Equation (3.1) vanishes.

Hence, Equation (3.1) becomes

$$\partial(\sigma(\vartheta)) = \sum_{b\in B}(-1)^{\dim b}x_b,$$

so $\sigma(\vartheta)$ is an Euler chain in B.

The map σ, from the set of Euler chains in A to the set of Euler chains in B, is equivariant for the action of $\mathrm{Z}_1(A;\mathbb{Z}) = \mathrm{Z}_1(B;\mathbb{Z})$ (recall that the action is by addition). Hence the quotient map

$$\sigma : \mathrm{Eul}(A) \to \mathrm{Eul}(B)$$

is equivariant for the action of $\mathrm{H}_1(A;\mathbb{Z}) = \mathrm{H}_1(B;\mathbb{Z})$, so it is a bijection. □

Remark. The bijectivity of σ follows from the equivariance only if $\mathrm{Eul}(A) \neq \emptyset$. However, $\mathrm{Eul}(A) = \emptyset$ if and only if $\mathrm{Eul}(B) = \emptyset$ (since both conditions are equivalent to $\chi(A) \neq 0$), so also in this degenerate case the concepts of Euler structure on A and on B are equivalent.

3.1.2 Smooth Euler structures

It is well known that the existence of a non-vanishing tangent vector field on a manifold W is equivalent to the vanishing of the Euler characteristic of W. In case $\chi(W) = 0$, it is easy to check that two non-vanishing tangent vector fields on W are not always homotopic through a non-vanishing homotopy. Even relaxing the condition and admitting that two vector fields are equivalent if they are homotopic (through a non-vanishing homotopy) out of an open ball $B \subseteq W$, it is generally false that two arbitrary vector fields on W are equivalent. As it is shown in this section, the equivalence classes of vector fields are in a bijective correspondence with the Euler structures on W (once a CW structure of W is chosen).

Definition 3.4. Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$. Two non-vanishing tangent vector fields v and w are **homologous** is there exists an open ball $B^n \subseteq W$ such that v and w are homotopic on $W \setminus B^n$ through a non-vanishing homotopy.

Remark. As W is connected, there always exists an isotopy that carries any ball $B^n \subseteq W$ to any other, so the relation of homology between tangent vector fields is an equivalence relation.

Definition 3.5. Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$. Each homology class of tangent vector field is called (**smooth**) **Euler structure**. The set of all smooth Euler structures is denoted by $\mathrm{vect}(W)$.

Endow W with a Riemannian metric. By rescaling, a non-vanishing tangent vector field can be thought of as a smooth section of the S^{n-1}-bundle $SW \to W$ of unit tangent vectors. In the same way a non-vanishing homotopy of tangent vector fields can be thought as a homotopy of sections of the bundle $SW \to W$.

Now choose a CW structure of W such that there exists a unique n-cell e^n (such a CW structure exists since W is connected). If v and w are two normal vector fields, there exists a class $d(v, w) \in \mathrm{H}^{n-1}(W; \mathbb{Z})$, called **primary difference**, such that v and w are homotopic out of e_n (i.e. on the $(n-1)$-skeleton of W) if and only if $d(v, w)$ vanishes (cf. [Ste51, §36]). Since the primary difference does not depend on the choice of the CW structure by [Ste51, Corollary 36.10], the n-dimensional cell can be chosen arbitrarily. This means that in fact $d(v, w)$ is the obstruction to the homology of v and w. This observation is summed up in the following lemma.

Lemma 3.6. *Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$ and with a given CW structure.*

Two non-vanishing vector fields v and w on W are homologous if and only if they are homotopic on W^{n-1} through a non-vanishing homotopy, that is if and only if their primary difference $d(v, w)$ vanishes.

Another useful lemma from Obstruction Theory is the following addition formula.

Lemma 3.7 (Addition formula, [Ste51, §36.6])**.** *If W is an n-dimensional closed connected manifold with $n \geq 2$ and $\chi(W) = 0$, and v, w and z are three non-vanishing vector fields, then*

$$d(v, z) = d(v, w) + d(w, z).$$

Theorem 3.8. *Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$.*

There exists a free and transitive action of $\mathrm{H}_1(W; \mathbb{Z})$ *on* $\mathrm{vect}(W)$*, which is defined as follows: for each Euler structure $[v]$ and for all $h \in \mathrm{H}_1(W, \mathbb{Z})$, the Euler structure $h \cdot [v]$ is represented by a vector field w such that*

$$\mathrm{PD}(d(v, w)) = h.$$

As $\chi(W) = 0$, at least one smooth Euler structure on W exists, so $\mathrm{vect}(W)$ *is an* $\mathrm{H}_1(W, \mathbb{Z})$*-affine space.*

Proof. First, if v is a normal vector field and $h \in \mathrm{H}_1(W; \mathbb{Z})$, then by [Ste51, §37.2] there exists a normal vector field w such that $\mathrm{PD}(d(v, w)) = h$.

Let us check the good definition of the action. If w_1 and w_2 are two normal vector fields such that $\mathrm{PD}(d(v, w_1)) = \mathrm{PD}(d(v, w_2)) = h$, then, by the addition formula (cf. Lemma 3.7) $d(w_1, w_2) = 0$, so w_1 and w_2 define the same smooth Euler structure. Analogously, if $[v_1] = [v_2]$, then by Lemma 3.6 $d(v_1, v_2) = 0$. Hence, for each normal vector field w, if $\mathrm{PD}(d(v_1, w)) = h$, so is $\mathrm{PD}(d(v_2, w))$ (cf. Lemma 3.7).

Thus, there exists a well defined map

$$\mathrm{vect}(W) \times \mathrm{H}_1(W; \mathbb{Z}) \to \mathrm{vect}(W),$$

which is an action due to the addition formula (cf. Lemma 3.7).

The action is obviously transitive because $\forall\, [v], [w] \in \mathrm{vect}(W)$ the homology class $\mathrm{PD}(d(v, w))$ carries v to w.

Finally, if $h \cdot [v] = [v]$, then v represents the Euler structure $h \cdot [v]$, so $h = d(v, v) = 0$. Hence the action is also free. □

The action of $\mathrm{H}_1(W; \mathbb{Z})$ on $\mathrm{vect}(W)$ has also a more geometrical interpretation, which is explained in [Tur90, §5.2]. Let v be a normal vector field on W, and let h be a homology class in $\mathrm{H}_1(W; \mathbb{Z})$. Let l be a simple closed

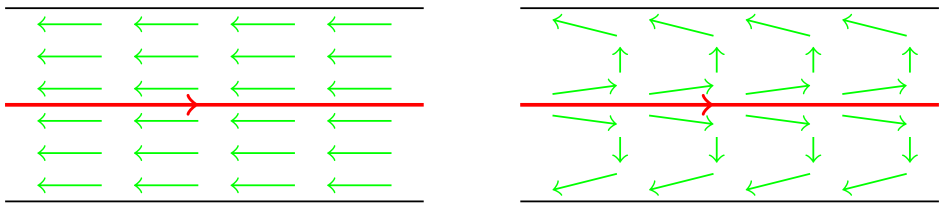


Figure 3.1: The picture shows the effect of the Reeb turbulization of the vector field represented by the green arrows along the red curve.

curve in W representing h. Choose a tubular neighbourhood U of l, and denote with e_t the tangent vector field on l, which can be extended on U using the product structure of the neighbourhood, and with e_r the normal outward radial vector field, which is defined on $U \setminus l$. We may assume that the vector field v coincides with $-e_t$ on U (changing v with a homologous vector field). The Euler structure $h \cdot [v]$ is then represented by the vector field obtained by gluing $v_{|W \setminus U}$ and $(\cos(\pi r/R) \cdot e_t + \sin(\pi r/R) \cdot e_r)_{|U}$, where r is the function assigning the distance from l and R is the radius of the tubular neighbourhood U. The resulting vector field is called **Reeb turbulization** of v along l.

Let us explain why the Reeb turbulization is the action of $H_1(W;\mathbb{Z})$ on $\mathrm{vect}(W)$. We have to check that, if the vector field v is the turbulization of u along the curve l, then $d(u,v) = \mathrm{PD}^{-1}([l])$. First, choose a cellular decomposition of W such that the regular neighbourhood U in which the field is changed is made up of one $(n-1)$-cell $\overline{e}$ and one n-cell $\overline{f}$ (cf. Figure 3.2). Orient $\overline{e}$ in such a way that the intersection between l and $\overline{e}$ is positive. The class $d(u,v)$ is then represented by the $(n-1)$-dimensional cellular cochain φ whose value on the cell $\overline{e}$ is 1 and on each other $(n-1)$-cell is 0 (cf. [DK01, Ch. 7]). φ is a cocycle because in the boundary of each n-cell f different from $\overline{f}$ the $(n-1)$-cell $\overline{e}$ does not appear, so

$$\langle \delta\varphi, f \rangle = \langle \varphi, \partial f \rangle = 0,$$

whereas in the boundary of $\overline{f}$ the $(n-1)$-cell $\overline{e}$ appear twice, with opposite orientations, hence

$$\langle \delta\varphi, \overline{f} \rangle = \langle \varphi, \overline{e} \rangle - \langle \varphi, \overline{e} \rangle + 0 = 0.$$

The cocycle φ is such that for each $(n-1)$-cell e

$$\langle \varphi, e \rangle = \#(l \cap e).$$

Therefore, it is the Poincaré dual of $[l] \in \mathrm{H}_1(W;\mathbb{Z})$ (cf. [Bre93, Ch. VI, Sect. 6]).

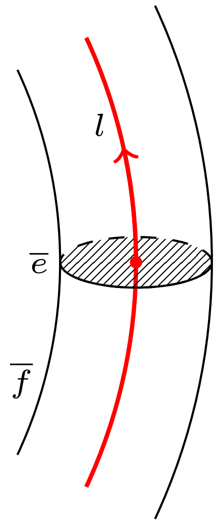


Figure 3.2: The cellular structure of the neighbourhood U of l.

Equivalence with combinatorial Euler structures

If W is a triangulated manifold, then there is an equivalence between combinatorial Euler structures and smooth Euler structures on W, which, in a way that will be explained, does not depend on the triangulation of W.

Theorem 3.9 ([Tur90, Lemma 6.3.4])**.** *Let W be an n-dimensional closed connected manifold, and let (A, ρ_A) be a smooth triangulation of W (i.e. A is a simplicial complex and $\rho_A : A \to W$ is a homeomorphism that is a C^∞ map on every simplex).*

Then there exist a bijection

$$(\rho_A)_\square : \mathrm{Eul}(A) \longrightarrow \mathrm{vect}(W)$$

equivariant for the action of $\mathrm{H}_1(A;\mathbb{Z}) \cong \mathrm{H}_1(W;\mathbb{Z})$, i.e. $\forall\, h \in \mathrm{H}_1(A,\mathbb{Z})$

$$(\rho_A)_\square(h \cdot [\vartheta]) = (\rho_A)_*(h) \cdot (\rho_A)_\square([\vartheta]).$$

The proof of this result will require the rest of the subsection.

First, a vector field on W will be constructed starting from a combinatorial Euler structure. Such a vector field will be a turbulization of a certain singular vector field associated to the simplicial structure, which will be denoted by F_1.

Let (A, ρ_A) be a simplicial structure of W. Let A' be the first barycentric subdivision of A: each p-simplex of A' is denoted by a $(p+1)$-tuple of simplices in A $\langle a_0, \ldots, a_p \rangle$, where a_i is an i-simplex and $a_i \subseteq a_{i+1}$. The simplex in A' denoted by $\langle a_0, \ldots, a_p \rangle$ is the simplex whose vertices are the centers $x_{a_0}, \ldots, x_{a_p}$ of the simplices in A $a_0, \ldots, a_p$ (cf. Figure 3.3).

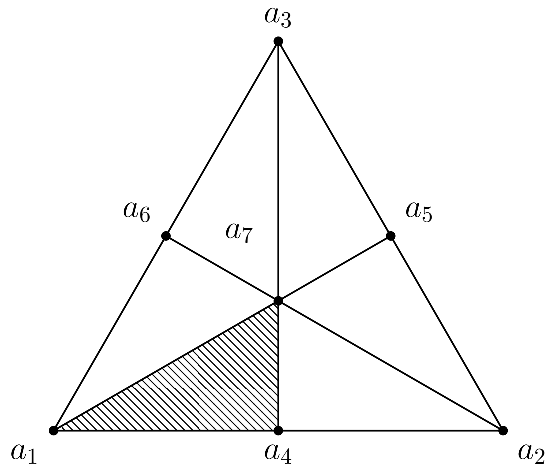


Figure 3.3: The picture shows the barycentric subdivision of a 2-simplex of a triangulation. The coloured simplex is the simplex denoted by $\langle a_1, a_4, a_7\rangle$.

The **field** F_1 **associated to the triangulation** A is defined as follows: if $y \in \langle a_0, \dots, a_p\rangle$, and the barycentric coordinates of y with respect to $x_{a_0}, \dots, x_{a_p}$ are denoted by $\lambda_0(y), \dots, \lambda_p(y)$, then

$$F_1(y) = \sum_{0\le i<j\le p} \lambda_i(y)\lambda_j(y)(x_{a_j} - y). \tag{3.2}$$

Note that for the sake of simplicity the simplex a_p and its image through ρ_A are identified, as well as the vector fields on abstract simplices and on W are identified through $\mathrm{d}(\rho_A)$.

The field F_1 is well defined on the whole W (the restriction of F_1 defined on a certain p-simplex $\langle a_0, \dots, a_p\rangle$ to a $(p-1)$-simplex on its boundary is the same as the field F_1 defined on the $(p-1)$-simplex).

It is easy to check that the singular points of F_1 on a simplex $\langle a_0, \dots, a_p\rangle$ are exactly the centers of the simplices x_{a_j}. Indeed, let y be a zero for F_1. Let k be smallest integer such that $\lambda_k(y) \neq 0$. Then it is easy to check that the vectors $(x_{a_j} - y)_{j>k}$ are independent. Hence, by Equation (3.2), $F_1(y) = 0$ implies that

$$(\lambda_k(y) + \lambda_{k+1}(y) + \ldots + \lambda_{j-1}(y))\,\lambda_j(y) = 0 \quad \forall\, k \le i < j \le n.$$

Since $\lambda_i(y) \ge 0\ \forall\, i$ and $\lambda_k(y) > 0$, it follows that

$$\lambda_j(y) = 0 \quad \forall\, j > k.$$

As k was the smallest index such that $\lambda_k(y) \neq 0$, it follows that $\lambda_j(y) = 0$ $\forall\, j \neq k$, so y must be x_{a_k}.

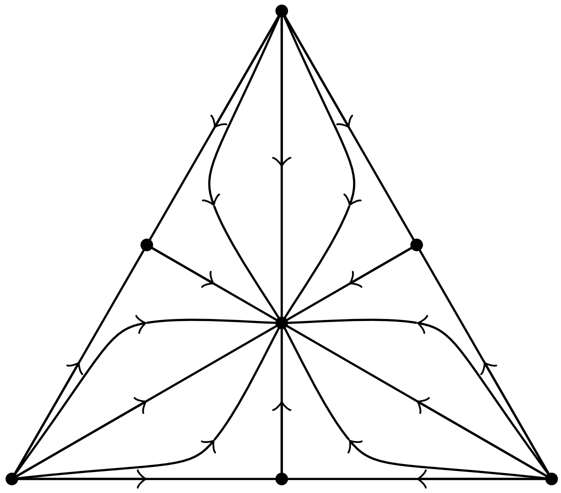


Figure 3.4: The picture shows how the vector field F_1 looks like on an abstract simplex a of the triangulation A. The singular points of this vector field are exactly the 0-simplices of the barycentric subdivision A'.

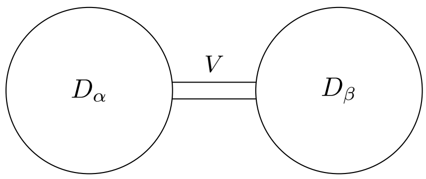


Figure 3.5: Preliminary definitions for the proof of Theorem 3.9.

To have a geometric idea of how F_1 appears on a single abstract simplex, see Figure 3.4.

In order to define the map $\mathrm{Eul}(A) \to \mathrm{vect}(W)$, some preliminary definitions are necessary. I will use the same notation as [Tur90, §6.2].

Let $\alpha = (-3/2, 0, \dots, 0) \in \mathbb{R}^n$ and $\beta = (3/2, 0, \dots, 0) \in \mathbb{R}^n$. Let D_α (resp. D_β) be the closed ball in $\mathbb{R}^n$ centered at α (resp. β) with radius 1. Let V be the set of points with distance less or equal to $1/10$ from the segment $[\alpha, \beta]$ that do not belong to $\mathring{D}_\alpha \cup \mathring{D}_\beta$ (cf. Figure 3.5). Let then B_α (resp. B_β) denote $V \cap D_\alpha$ (resp. $V \cap D_\beta$).

A **special vector field** on V is a field of non-vanishing vectors tangent to $\mathbb{R}^n$ on V with the property that the vectors on $\partial V \setminus (B_\alpha \cup B_\beta)$ are collinear (including the orientation) with $\beta - \alpha$.

Let u be a special vector field on V. Then, the map $B_\alpha \to \partial D_\alpha$ defined by $x \mapsto \alpha + u(x)/\|u(x)\|$ factors through $B_\alpha \to B_\alpha/\partial B_\alpha$. The quotient map of spheres $B_\alpha/\partial B_\alpha \to \partial D_\alpha$ has a well-defined degree (since if the orientation of D_α is changed, the orientation of $B_\alpha/\partial B_\alpha$ automatically changes), which will be denoted by $\mathrm{ind}_{\alpha,\beta}(u)$.

Endow ∂D_α and ∂D_β with the same orientation (up to translation). Then orientations on $B_\alpha/\partial B_\alpha$ and $B_\beta/\partial B_\beta$ are induced. Since u is non-vanishing on V, the maps $B_\alpha/\partial B_\alpha \to \partial D_\alpha$ and $B_\beta/\partial B_\beta \to \partial D_\beta$ have opposite degree (roughly speaking, the latter is the former composed with a reflection along the axis defined by $\beta - \alpha$), that is $\mathrm{ind}_{\alpha,\beta}(u) = -\mathrm{ind}_{\beta,\alpha}(u)$.

Lemma 3.10. *For any integer $i \in \mathbb{Z}$ there exists a special vector field u_i on V such that* $\mathrm{ind}_{\alpha,\beta}(u_i) = i$.

Proof. This lemma is a straightforward consequence of the existence of maps of spheres $S^{n-1} \to S^{n-1}$ of any degree.

Let $f : D_\alpha^{n-1} \to S^{n-1}$ be a map such that on the boundary ∂D_α^{n-1} it is constant and collinear with $\beta - \alpha$, and the degree of the quotient map $S^{n-1} \to S^{n-1}$ is i. Extend f on V in such a way that it is constant in the first component. The result will be a special vector field u_i such that $\mathrm{ind}_{\alpha,\beta}(u_i) = i$. □

Proof of Theorem 3.9. Let ξ be an Euler structure on the simplicial complex A. ξ can be represented as a simplicial 1-chain ϑ in A', the first barycentric subdivision of A:

$$\vartheta = \sum_{\langle a,b\rangle \in A'} \vartheta(a,b) \cdot \langle a,b\rangle.$$

For each vertex a of A', let $D(a) \subseteq W$ denote a small closed ball centered in $\rho_{A'}(a)$, so that the balls $D(a)$ are all disjoint. Moreover, for each 1-simplex $\langle a,b\rangle$ in A', let $V(a,b)$ denote a closed tubular neighbourhood of $p_{A'}(\langle a,b\rangle)$ to which the intersection with $D(a)$ and $D(b)$ are removed. $D(a)$, $D(b)$ and $V(a,b)$ appear as D_α, D_β and V in Figure 3.5.

Consider F_1, the singular vector field associated to the triangulation A, defined by Equation (3.2). As its singular points are the vertices of A', F_1 is non-vanishing on $W \setminus \coprod D(a)$. Up to a small isotopy, F_1 appears on $V(a,b)$ as the vector field u_0 (otherwise it does on $V(b,a)$). Now focus on a specific 1-chain $\langle a,b\rangle$, and suppose that F_1 appears as u_0 on $V(a,b)$. By Lemma 3.10, there exists a special vector field $u_{\vartheta(a,b)}$ on $V(a,b)$ such that it coincides with F_1 on $\partial V(a,b)$ and

$$\mathrm{ind}_{a,b}(u_{\vartheta(a,b)}) = \vartheta(a,b). \tag{3.3}$$

Let F_ϑ denote the field obtained attaching F_1 outside $\coprod D(a) \cup \coprod V(a,b)$ and $u_{\vartheta(a,b)}$ on each $V(a,b)$. It is clear that F_ϑ is a non-vanishing vector field defined on $W \setminus \coprod D(a)$.

Now we will show that in fact F_ϑ extends to a non-vanishing vector field on W, and it will be the image of $[\vartheta]$ in $\mathrm{vect}(W)$.

Let a be a specific vertex of the triangulation A'. Since the degree

$$\deg : \pi_{n-1}(S^{n-1}) \longrightarrow \mathbb{Z}$$

is an isomorphism, the field F_ϑ extends on $D(a)$ if and only if the degree of the map $\partial D(a) \to S^{n-1}$ defined by $x \mapsto F_\vartheta(x)/\|F_\vartheta(x)\|$ has degree equal to 0. Let g denote this map, and let $g_0 : \partial D(a) \to S^{n-1}$ denote the map defined by $x \mapsto F_1(x)/\|F_1(x)\|$.

Since $F_{\vartheta|_{\partial D(a)}}$ is obtained changing $F_{1|_{\partial D(a)}}$ on $\coprod \overline{V(a,b)} \cap \partial D(a)$, it follows that

$$\deg g = \deg g_0 + \sum_b \left(\mathrm{ind}_{a,b}(F_\vartheta) - \mathrm{ind}_{a,b}(F_1)\right),$$

where the sum runs over the vertices b of A' next to a. Since F_1 is a small perturbation of a constant vector field on $\partial D(a) \cap \overline{V(a,b)}$, the index $\mathrm{ind}_{a,b}(F_1)$ vanishes for each i, hence

$$\deg g = \deg g_0 + \sum_b \mathrm{ind}_{a,b}(F_\vartheta).$$

Recall now that a and b are simplices in A, and either $a \subseteq b$ or $b \subseteq a$. The field F_1 is directed towards the simplex in A whose dimension is the biggest among $\dim a$ and $\dim b$. Hence, if $a \subseteq b$ then F_1 is like u_0, whereas if $b \subseteq a$ then F_1 is like $-u_0$. Now using the fact that if F_1 is like u_0 then $\mathrm{ind}_{a,b}(F_\vartheta) = \vartheta(a,b)$ (cf. Equation (3.3)), we have that

$$\begin{aligned}
\deg g &= \deg g_0 + \sum_{b \supseteq a} \mathrm{ind}_{a,b}(F_\vartheta) + \sum_{b \subseteq a} \mathrm{ind}_{a,b}(F_\vartheta) \\
&= \deg g_0 + \sum_{b \supseteq a} \vartheta(a,b) - \sum_{b \subseteq a} \mathrm{ind}_{b,a}(F_\vartheta) \\
&= \deg g_0 + \sum_{b \supseteq a} \vartheta(a,b) - \sum_{b \subseteq a} \vartheta(b,a),
\end{aligned}$$

which, thanks to the equality $\deg g_0 = (-1)^{\dim a}$, implies

$$\deg g = (-1)^{\dim a} + \sum_{b \supseteq a} \vartheta(a,b) - \sum_{b \subseteq a} \vartheta(b,a). \tag{3.4}$$

Since

$$\begin{aligned}
\sum_{c\in A}(-1)^{\dim c}\cdot c = \partial\vartheta \\
&= \partial\left(\sum_{\langle c,d\rangle\in A'}\vartheta(c,d)\cdot\langle c,d\rangle\right)\\
&= \partial\left(\sum_{\substack{c,d\in A\\ c\subseteq d}}\vartheta(c,d)\cdot\langle c,d\rangle\right)\\
&= \sum_{\substack{c,d\in A\\ c\subseteq d}}\vartheta(c,d)\cdot(d-c)\\
&= \sum_{c\in A}\left(\sum_{d\subseteq c}\vartheta(d,c)-\sum_{d\supseteq c}\vartheta(c,d)\right)\cdot c,
\end{aligned}$$

for each $c\in A$

$$(-1)^{\dim c}=\sum_{d\subseteq c}\vartheta(d,c)-\sum_{d\supseteq c}\vartheta(c,d),$$

hence Equation (3.4) proves that $\deg g=0$ and that, therefore, F_ϑ extends to a non-vanishing field on $D(a)$. Note that such an extension is unique up to homotopy because $D(a)$ is contractible.

As a was arbitrary, F_ϑ extends to a non-vanishing vector field on the whole W. For a combinatorial Euler chain ϑ, we define

$$(\rho_A)_\triangle(\vartheta)=[F_\vartheta]\in\operatorname{vect}(W). \tag{3.5}$$

It is noteworthy that in fact $[F_\vartheta]$ does not depend on the choice of $u_{\vartheta(a,b)}$ on each $V(a,b)$, since $D(a)\cup D(b)\cup V(a,b)$ is a ball, and, by definition, changing F_ϑ inside a ball does not change its class in $\operatorname{vect}(W)$.

The map $(\rho_A)_\triangle$ is equivariant for the action of $\mathrm{Z}_1(A;\mathbb{Z})\to\mathrm{H}_1(W;\mathbb{Z})$, i.e. for each simplicial one-dimensional cycle γ

$$(\rho_A)_\triangle(\gamma+\vartheta)=[\gamma]\cdot[F_\vartheta]\,.$$

Now we will prove this fact. Consider $\gamma\in\mathrm{Z}(A',\mathbb{Z})$:

$$\gamma=\sum_{a\subseteq b}\gamma(a,b)\cdot\langle a,b\rangle.$$

The singular cycle associated to γ on the manifold W can be constructed by taking in each $V(a,b)$ $|\gamma(a,b)|$ oriented segments, going from a to b if $\gamma(a,b)\geq 0$, and going from b to a if $\gamma(b,a)<0$. The disjoint union of the segments extends to a singular cycle by adding arcs in $\coprod D(a)$ because $\partial\gamma=$

0. The obtained singular cycle c is a disjoint union of oriented simple closed curves $c_1, \dots, c_s$. After changing F_ϑ by homotopy so that F_ϑ is parallel to c in a small neighbourhood of c, perform a Reeb turbulization on each c_i. Denote by F' the result of the turbulization (so $[F'] = [\gamma] \cdot [F_\vartheta]$). It is easy to check that

$$\mathrm{ind}_{a,b}(F') = \mathrm{ind}_{a,b}(F_\vartheta) + \gamma(a,b) = \vartheta(a,b) + \gamma(a,b).$$

Hence $[F']$ is $(\rho_A)_\triangle(\vartheta+\gamma)$ (cf. Equations (3.3) and (3.5)), so the map $(\rho_A)_\triangle$ is equivariant for the action of $\mathrm{Z}_1(A;\mathbb{Z}) \to \mathrm{H}_1(W;\mathbb{Z})$.

This implies at once that $(\rho_A)_\triangle$ passes to a quotient map

$$(\rho_A)_\square : \mathrm{Eul}(A) \to \mathrm{vect}(W)$$

and that this map is equivariant for the action of $\mathrm{H}_1(A;\mathbb{Z}) \cong \mathrm{H}_1(W;\mathbb{Z})$. $\square$

We proved that the set of combinatorial Euler structures and the set of smooth Euler structures can be identified for a triangulated manifold W. The next step consists of proving that the identification does not depend on the triangulation, in the sense of Theorem 3.11.

Let (A, ρ_A) and (B, ρ_B) be two smooth triangulations of a closed connected n-manifold W. The homeomorphism $\rho_B^{-1} \circ \rho_A$ is homotopic to a piecewise linear homomorphism $f_{A,B}$ (cf. [Mun66]). Hence there exist simplicial subdivisions A' and B' of A and B such that $f_{A,B}$ is a simplicial isomorphism of A' and B'.

If $\sigma_{A,A'} : \mathrm{Eul}(A) \to \mathrm{Eul}(A')$ and $\sigma_{B;B'} : \mathrm{Eul}(B) \to \mathrm{Eul}(B')$ are the isomorphisms given by Theorem 3.3, then

$$\sigma_{B,B'}^{-1} \circ (f_{A,B})_* \circ \sigma_{A,A'} : \mathrm{Eul}(A) \longrightarrow \mathrm{Eul}(B) \tag{3.6}$$

is a bijection equivariant for the action of $\mathrm{H}_1(A;\mathbb{Z}) \cong \mathrm{H}_1(B;\mathbb{Z})$, and it does not depend on the choice of the piecewise linear approximation and of the simplicial refinements A' and B'. This map will be denoted by

$$\tau_{A,B} : \mathrm{Eul}(A) \longrightarrow \mathrm{Eul}(B). \tag{3.7}$$

Theorem 3.11 ([Tur90, Theorem 6.1.2])**.** *Let (A, ρ_A) and (B, ρ_B) be two smooth triangulations of an n-dimensional closed connected manifold W.*

Then, Diagram 3.1 is commutative.

Proof. If B is a simplicial refinement of A, then, according to [Tur90, Lemma 6.4.1], there exist triangulations $A = A_1, \dots, A_t = B$ such that for each i the triangulations A_i and A_{i+1} coincide outside the star of some simplex (recall that the star of a simplex a is the union of all simplices intersecting a). Since the image of a simplex is contained in a ball in W, Diagram 3.2 is commutative for each i ($\sigma_{A_i, A_{i+1}}$ is the map given by Theorem 3.3).

$$\begin{tikzcd}
\mathrm{Eul}(A) \arrow[rr, "\tau_{A,B}"] \arrow[dr, swap, "(\rho_A)_\square"] & & \mathrm{Eul}(B) \arrow[dl, "(\rho_B)_\square"] \\
& \mathrm{vect}(W) &
\end{tikzcd}$$

Diagram 3.1: Commutativity of the maps $(\rho_A)_\square$ and $(\rho_B)_\square$.

$$\begin{tikzcd}
\mathrm{Eul}(A_i) \arrow[rr, "\sigma_{A_i,\, A_{i+1}}"] \arrow[dr, swap, "(\rho_{A_i})_\square"] & & \mathrm{Eul}(A_{i+1}) \arrow[dl, "(\rho_{A_{i+1}})_\square"] \\
& \mathrm{vect}(W) &
\end{tikzcd}$$

Diagram 3.2: Commutativity of the maps $(\rho_{A_i})_\square$ and $(\rho_{A_{i+1}})_\square$.

Hence Diagram 3.1 is commutative too (note that, since B is a refinement of A, $\sigma_{A,B} = \tau_{A,B}$). The theorem is therefore proved if B is a refinement of A.

If now A and B are arbitrary triangulations, there exist simplicial refinements A' and B' and a chain of triangulations $(A', \eta_1), \ldots, (A', \eta_s)$ such that $\eta_1 = \rho_{A'}$ and, for each i, η_i and η_{i+1} are homotopic and coincide outside $\eta_i^{-1}(B^n)$ for some ball $B^n \subseteq W$, and there exists a simplicial isomorphism $\psi : A' \to B'$ such that $\rho_{B'} = \eta_s \circ \psi^{-1}$ (cf. [Mun66]).

Since η_i and η_{i+1} coincide outside the preimage of some ball in W, it follows that $(\eta_i)_\square = (\eta_{i+1})_\square$, so $(\rho_{A'})_\square = (\eta_1)_\square = (\eta_s)_\square$.

Consider Diagram 3.3.

$$\begin{tikzcd}
\mathrm{Eul}(A) \arrow[r, "\tau_{A,B}"] \arrow[d, "\sigma_{A,\, A'}"] \arrow[dd, bend right=60, swap, "(\rho_A)_\square"] & \mathrm{Eul}(B) \arrow[d, "\sigma_{B,\, B'}"] \arrow[dd, bend left=60, "(\rho_B)_\square"] \\
\mathrm{Eul}(A') \arrow[r, "\psi_*"] \arrow[d, "(\rho_{A'})_\square"] \arrow[dr, "(\eta_s)_\square"] & \mathrm{Eul}(B') \arrow[d, "(\rho_{B'})_\square"] \\
\mathrm{vect}(W) \arrow[r, "\mathrm{id}"] & \mathrm{vect}(W)
\end{tikzcd}$$

Diagram 3.3: Proof of the commutativity of the maps $(\rho_A)_\square$ and $(\rho_B)_\square$.

The triangles on the left and on the right commute because the theorem is already proved in the case one of the two triangulations is a refinement of the other. The top square commutes because of the definition of $\tau_{A,B}$ (cf, Equations (3.6) and (3.7)). The left bottom triangle is commutative because $(\rho_{A'})_\square = (\eta_s)_\square$, whereas the right bottom triangle is commutative because $\rho_{B'} = \eta_s \circ \psi^{-1}$.

Hence Diagram 3.3 is commutative, and this proves the commutativity of Diagram 3.1. $\square$

3.1.3 Normal Euler structures

There is a third way of defining Euler structures, which is explained also in [Tur97]. Let W be an oriented n-manifold. Then each fiber S_xW of the sphere bundle of normal tangent vectors $\pi : SW \to W$ is endowed with a choice of the generator of $\mathrm{H}^{n-1}(S_xW)$.

Definition 3.12. Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$. A **normal Euler structure** on W is a cohomology class in $\mathrm{H}^{n-1}(SW)$ that restricts on each fiber as the standard generator of $\mathrm{H}^{n-1}(S_xW)$.

The set of normal Euler structures on W is denoted by $\mathrm{nvect}(W)$.

Lemma 3.13. $\mathrm{nvect}(W)$ *is endowed with a free and transitive action of* $\mathrm{H}_1(W;\mathbb{Z}) \cong \mathrm{H}^{n-1}(W;\mathbb{Z})$, *which is given by the pullback map*

$$\pi^* : \mathrm{H}^{n-1}(W;\mathbb{Z}) \longrightarrow \mathrm{H}^{n-1}(SW;\mathbb{Z})$$

composed with addition, i.e. for each $h \in \mathrm{H}_1(W;\mathbb{Z})$ *and for each* $c \in \mathrm{nvect}(W)$

$$h \cdot c = c + \pi^*(\mathrm{PD}^{-1}(h)).$$

Proof. $h \cdot c$ is still a normal Euler structure. Indeed, the restriction to the fiber S_xW is

$$i_x^*(h \cdot c) = i_x^*(c + \pi^*(\mathrm{PD}^{-1}(h))) = i_x^*(c) + i_x^*(\pi^*(\mathrm{PD}^{-1}(h))), \tag{3.8}$$

where i_x is the embedding of S_xW in SW. $i_x^* \circ \pi^*$ is 0 because the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "i_x"] & \{x\} \arrow[d] \\ SW \arrow[r, "\pi"] & W \end{tikzcd}$$

commutes and the cohomology group $\mathrm{H}^{n-1}(\{x\},\mathbb{Z})$ vanishes. Hence, Equation (3.8) implies that

$$i_x^*(h \cdot c) = i_x^*(c),$$

and therefore $h \cdot c$ is still a normal Euler structure.

The action of $\mathrm{H}_1(W;\mathbb{Z})$ on $\mathrm{nvect}(W)$ is free and transitive due to the Leray-Hirsch Theorem.

Theorem 3.14 (Leray-Hirsch, [Hat02, Theorem 4D.1])**.** *Let $\pi : E \to W$ be a fiber bundle with fiber F such that, for some commutative coefficient ring R:*

1. *$\mathrm{H}^n(F;R)$ is a finitely generated free R-module for each n.*
2. *There exist classes $c_j \in \mathrm{H}^{k_j}(E;R)$ whose restrictions $i^*(c_j)$ form a basis for $\mathrm{H}^*(F;R)$ in each fiber F, where $i : F \to E$ is the inclusion.*

Then the map

$$\begin{aligned} \Phi : \quad \mathrm{H}^*(W;R) \otimes_R \mathrm{H}^*(F;R) &\longrightarrow \mathrm{H}^*(E;R) \\ \sum_{i,j} b_i \otimes i^*(c_j) &\longmapsto \sum_{i,j} \pi^*(b_i) \smile c_j \end{aligned}$$

is an isomorphism.

Apply Theorem 3.14 to the bundle $SW \to W$ with coefficient in $\mathbb{Z}$. The class $1 \in \mathrm{H}^0(SW;\mathbb{Z})$ and a normal Euler structure $c \in \mathrm{H}^{n-1}(SW;\mathbb{Z})$ satisfy the hypothesis of the theorem (the existence of normal Euler structure is proved in Corollary 3.16).

Normal Euler structures are identified through the map Φ^{-1} with the subset of $\mathrm{H}^*(W;\mathbb{Z}) \otimes_R \mathrm{H}^*(F;\mathbb{Z})$ consisting of the elements

$$h \otimes 1 + 1 \otimes i^*c,$$

for some $h \in \mathrm{H}^{n-1}(W;\mathbb{Z})$. $h' \in \mathrm{H}^{n-1}(W;\mathbb{Z})$ acts on this set by addition of $h' \otimes 1$, so the action is clearly free and transitive. □

Equivalence with smooth Euler structures

Smooth Euler structures and normal Euler structures are in fact the same, as Theorem 3.15 shows.

Theorem 3.15. *Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$.*

Then, there exists an $\mathrm{H}_1(W;\mathbb{Z})$-equivariant bijection

$$\tau : \mathrm{vect}(W) \longrightarrow \mathrm{nvect}(W).$$

Proof. Let u be a non-vanishing vector field on W. Without loss of generality we may assume that u is a normal vector field, that is a section $u : W \to SW$. The image of u defines an n-cycle in SW, which, by Poincaré duality, defines a class in $\mathrm{H}^{n-1}(SW;\mathbb{Z})$, which will be $\tau(u)$. The cycle $\operatorname{Im} u$ satisfies

$$\#(\operatorname{Im} u \cap S_x W) = +1$$

for each $x \in W$. This means that, if $i_x : S_x W \to SW$ is the immersion, $i_x^{-1}(\operatorname{Im} u)$ is a point in $S_x W$, so its Poincaré dual is the generator g of $\mathrm{H}^{n-1}(S_x W;\mathbb{Z})$. Hence

$$g = \mathrm{PD}^{-1}\left(i_x^{-1}(\operatorname{Im} u)\right) = i_x^*\left(\mathrm{PD}^{-1}(\operatorname{Im} u)\right) = i_x^*(\tau(u)),$$

where the central equality is a well-known consequence of the functoriality of the Thom Isomorphism (cf. [BT82, §6, Poincaré Duality and the Thom Class, pag. 65-69]). Thus, $\tau(u)$ is a normal Euler structure.

Let us check that the map τ is well defined. Let u and v be normal vector fields that define the same Euler structure on W. Without loss of generality we may assume that u and v coincide out of some ball $B^n \subseteq W$. The manifold W can be endowed with a cellular structure such that B^n is an n-dimensional cell, so u and v coincide on the $(n-1)$-skeleton of W. By the description of Poincaré duality in [Bre93, Ch. VI, Sect. 6], the cohomology classes $\tau(u)$ and $\tau(v)$ are represented respectively by the cochains φ_u and φ_v such that for each $(n-1)$-cell e of W

$$\begin{aligned} \langle \varphi_u, e\rangle &= \#(\operatorname{Im} u \cap e), \\ \langle \varphi_v, e\rangle &= \#(\operatorname{Im} v \cap e). \end{aligned}$$

Since u and v coincide on the $(n-1)$-skeleton, the cochains φ_u and φ_v are in fact the same. Thus, $\tau(u) = \tau(v)$.

The last thing we have to prove is the $\mathrm{H}_1(W;\mathbb{Z})$-equivariance of τ. We will prove it in the special case of 3-manifolds (which is the case we will need). In this case the tangent bundle is trivial (cf. [Kir89, Ch. VII, Theorem 1]), so there exists a trivialization $SW \cong W \times S^2$. Let u be a normal vector field, and let $\widetilde{u}$ denote the turbulization of u along a curve l. We have to check that the classes $\tau(\widetilde{u}) - \tau(u)$ and $\mathrm{PD}^{-1}([l \times S^2])$ are the same class in $\mathrm{H}^2(SW;\mathbb{Z})$. By the description of Poincaré duality in [Bre93, Ch. VI, Sect. 6], $\tau(u)$ and $\tau(\widetilde{u})$ are represented respectively by the cochains φ_u and $\varphi_{\widetilde{u}}$ such that for each 2-cell e satisfy

$$\begin{aligned} \langle \varphi_u, e\rangle &= \#(\operatorname{Im} u \cap e), \\ \langle \varphi_{\widetilde{u}}, e\rangle &= \#(\operatorname{Im} \widetilde{u} \cap e), \end{aligned}$$

and analoguosly the cochain $\mathrm{PD}^{-1}([l \times S^2])$ satisfies for each 2-cell e

$$\langle \mathrm{PD}^{-1}([l \times S^2]), e\rangle = \#\left((l \times S^2) \cap e\right).$$

Thus, it is sufficient to prove that for any 2-cell e

$$\#(\operatorname{Im} \widetilde{u} \cap e) - \#(\operatorname{Im} u \cap e) = \#\left((l \times S^2) \cap e\right). \tag{3.9}$$

Let f_u and $f_{\widetilde{u}}$ denote the functions from W to S^2 that represent the vector fields u and $\widetilde{u}$. Up to changing the trivialization of SW we may assume that f_u is a constant f. Let $p \in S^2$ be a regular value of $f_{\widetilde{u}}$ different from f. Endow S^2 with the cellular structure consisting of the 0-cell p and a single 2-cell, and endow W with a cellular structure such that the regular neighbourhood U of l where u has been turbulized consists of one 2-cell $\overline{e}$ and one 3-cell $\overline{f}$ (as in Figure 3.2). The sphere bundle SW is then endowed with the product cellular structure. Note that the unique 2-cell in $U \times S^2 \subseteq W$ is $\overline{e} \times \{p\}$.

Let e be a 2-cell of $SW \cong W \times S^2$ different from $\overline{e} \times \{p\}$. e must be disjoint from $U \times S^2$, so

$$\#\left((l \times S^2) \cap e\right) = 0.$$

Moreover, since u and $\widetilde{u}$ coincide out of U, it follows that

$$\#(\operatorname{Im} \widetilde{u} \cap e) = \#(\operatorname{Im} u \cap e).$$

Thus, Equation (3.9) is satisfied for any e different from $\overline{e} \times \{p\}$.

Now focus on the 2-cell $\overline{e} \times \{p\}$. Since $\#(l \cap \overline{e}) = +1$, it follows that

$$\#\left((l \times S^2) \cap (\overline{e} \times \{p\})\right) = +1. \tag{3.10}$$

Since f_u is a constant different from p, $\operatorname{Im} u$ is disjoint from $\overline{e} \times \{p\}$, so

$$\#\left(\operatorname{Im} u \cap (\overline{e} \times \{p\})\right) = 0. \tag{3.11}$$

Finally, a point $(x, p) \in \overline{e} \times \{p\}$ belongs to $\operatorname{Im} \widetilde{u}$ if and only if $f_{\widetilde{u}}(x) = p$. Moreover, the sign of the intersection between $\operatorname{Im} \widetilde{u}$ and $\overline{e} \times \{p\}$ at the the point (x, p) is the determinant of the Jacobian $J_x(f_{\widetilde{u}})$. This proves that

$$\#\left(\operatorname{Im} \widetilde{u} \cap (\overline{e} \times \{p\})\right) = \deg g_{\widetilde{u}},$$

where $g_{\widetilde{u}} : S^2 \to S^2$ is obtained from the map $f_{\widetilde{u}} : \overline{e} \to S^2$ by quotienting the boundary of $\overline{e}$ to a point. By the definition of Reeb turbulization (cf. Figure 3.1) and the fact that g_u (the quotient of f_u) is a constant, we have that $\deg g_{\widetilde{u}} = +1$, so

$$\#\left(\operatorname{Im} \widetilde{u} \cap (\overline{e} \times \{p\})\right) = +1. \tag{3.12}$$

Equations (3.10), (3.11) and (3.12) prove that Equation (3.9) in the case e is the 2-cell $\overline{e} \times \{p\}$. Thus, $\tau(\widetilde{u}) - \tau(u)$ and $\mathrm{PD}^{-1}([l \times S^2])$ are the same class in $\mathrm{H}^2(SW; \mathbb{Z})$, so the map τ is $\mathrm{H}_1(W; \mathbb{Z})$-equivariant. □

Corollary 3.16. *Let W be a closed connected manifold of dimension $n \geq 2$ with $\chi(W) = 0$.*

Then, there exists a normal Euler structure on W.

Proof. The vanishing of Euler characteristic $\chi(W)$ is equivalent to the existence of a non-vanishing vector field, hence of a smooth Euler structure.

The existence of normal Euler structure is a consequence of the construction of a normal Euler structure starting from a smooth Euler structure, as in the proof of Theorem 3.15. □

3.2 Spin$^{\mathbb{C}}$ structures and Euler structures

In the case of 3-manifolds, the notion of Euler structures is actually equivalent to the one of Spin$^{\mathbb{C}}$ structure. In this section the equivalence between the definitions is proved. The proof is taken from [Tur97].

3.2.1 Spin$^{\mathbb{C}}$ structures on 3-manifolds

Let Y be a 3-manifold. There is another useful characterization of Spin$^{\mathbb{C}}$ structures. Before explaining it, note that the map

$$\begin{aligned} \mathrm{U}(1) \times \mathrm{SU}(2) &\longrightarrow \mathrm{U}(2) \\ (\lambda, A) &\longmapsto \lambda \cdot A \end{aligned}$$

induces an isomorphism

$$(\mathrm{U}(1) \times \mathrm{SU}(2))/\{\pm 1\} \cong \mathrm{U}(2).$$

Since Spin(3) is the universal cover of $\mathrm{SO}(3) = \mathbb{RP}^3$,

$$\mathrm{Spin}(3) \cong S^3 \cong \mathrm{SU}(2).$$

Hence Diagram 3.4 commutes.

$$\begin{tikzcd} \mathrm{Spin}^{\mathbb{C}}(3) \arrow[r, "\sim"] \arrow[dr] & (\mathrm{U}(1) \times \mathrm{SU}(2))/\{\pm 1\} \arrow[r, "\sim"] \arrow[d] & \mathrm{U}(2) \arrow[dl, dashed] \\ & \mathrm{SO}(3) & \end{tikzcd}$$

Diagram 3.4: The U(1)-principal bundle $\mathrm{Spin}^{\mathbb{C}}(3) \to \mathrm{SO}(3)$.

By Diagram 3.4 it is clear that the fiber bundle $\mathrm{U}(2) \to \mathrm{SO}(3)$ is actually a U(1)-principal bundle. Moreover, $\mathrm{U}(2) \to \mathrm{SO}(3)$ is a non-trivial bundle. In fact, it it were trivial, the fundamental group of the total space U(2) would be $\mathbb{Z} \times \mathbb{Z}_2$; however, the long exact sequence in homotopy of the fiber bundle $\mathrm{U}(1) \to \mathrm{U}(2) \to \mathrm{SO}(3)$ implies that $\pi_1(\mathrm{U}(2)) \cong \mathbb{Z}$ (cf. [Hat02, Theorem 4.40 and Proposition 4.47]), hence the bundle $\mathrm{U}(2) \to \mathrm{SO}(3)$ must be non-trivial.

By Lemma 1.34 and Lemma 1.41, there exists a bijective correspondence between the set of isomorphism classes of U(1)-principal bundles and $\mathrm{H}^2(\mathrm{SO}(3); \mathbb{Z}) \cong \mathbb{Z}_2$. The non-trivial bundle $\mathrm{U}(2) \to \mathrm{SO}(3)$ corresponds to the unique non-trivial element in $\mathbb{Z}_2$.

Definition 3.17. Let Y be a closed oriented connected 3-manifold, and let $\mathcal{P}_{\mathrm{SO}(3)}$ denote the SO(3)-principal bundle of orthonormal frames. The subset of $\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}, \mathbb{Z})$ of classes that restrict to the non-trivial element of $\mathrm{H}^2(\mathrm{SO}(3); \mathbb{Z})$ to each fiber is denoted by $\mathcal{S}(Y)$.

We will prove that $\mathcal{S}(Y)$ is in fact in bijection with the set of $\mathrm{Spin}^{\mathbb{C}}$ structures on Y. First, note that $\mathcal{S}(Y)$ is endowed with the action of $\mathrm{H}^2(Y; \mathbb{Z})$ given by the pullback $\mathrm{H}^2(Y; \mathbb{Z}) \to \mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z})$ and addition.

It is worth noting that, since Y is 3-dimensional and oriented, the SO(3)-principal bundle $\mathcal{P}_{\mathrm{SO}(3)}$ is the trivial fiber bundle $Y \times \mathrm{SO}(3)$ (cf. [Kir89, Ch. VII, Theorem 1]).

By the Künneth formula (cf. [Hat02, Theorem 3.15])

$$\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z}) \cong \mathrm{H}^2(Y; \mathbb{Z}) \oplus \mathbb{Z}_2.$$

The projection on the second component is simply the restriction homomorphism

$$\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{SO}(3); \mathbb{Z}) \cong \mathbb{Z}_2,$$

hence $\mathcal{S}(Y)$ is identified with the subset $\mathrm{H}^2(Y; \mathbb{Z}) \oplus \{1\}$ of $\mathrm{H}^2(Y; \mathbb{Z}) \oplus \mathbb{Z}_2$ (the classes that restrict to the non-trivial generator of $\mathbb{Z}_2$).

The pullback of a class $\alpha \in \mathrm{H}^2(Y; \mathbb{Z})$ to $\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z})$ is identified with $\alpha \oplus 0$ in $\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z})$. Note that in this form the action of $\alpha \in \mathrm{H}^2(Y; \mathbb{Z})$ is simply the addition of $\alpha \oplus 0$, so it is clear that the action is free and transitive.

Lemma 3.18. *There exists an $\mathrm{H}^2(Y; \mathbb{Z})$-equivariant bijection ν between the set of $Spin^{\mathbb{C}}$ structures on a closed oriented connected 3-manifold Y and $\mathcal{S}(Y)$.*

Proof. Let $(\eta : F \to Y, H)$ be a pair defining a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}$ on Y (cf. Definition 1.37). By [Kir89, Ch. VII, Theorem 1], $\mathcal{P}_{\mathrm{SO}(3)}$ is the trivial bundle, so it is induced by the trivial cocycle, i.e. the cocycle $(g_{\alpha\beta})$ on an acyclic cover (U_α) such that $g_{\alpha\beta} \equiv 1$ for each α and β.

By Lemma 1.38, the bundle $F \to Y$ is represented by a cocycle $(\widetilde{g}_{\alpha\beta})$ such that $\mathrm{Im}(g_{\alpha\beta})$ is included in the kernel of the projection

$$\mathrm{Spin}(3) \longrightarrow \mathrm{SO}(3),$$

which is isomorphic to U(1). Hence, F is also a U(1)-principal bundle over $\mathcal{P}_{\mathrm{SO}(3)}$. By Lemma 1.34 and Lemma 1.41, a class in $\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z})$ is associated to F. Such a class restricts to each fiber of $\mathcal{P}_{\mathrm{SO}(3)} \to Y$ to the non-zero element of $\mathrm{H}^2(\mathrm{SO}(3))$, because on each fiber the map $F \to \mathcal{P}_{\mathrm{SO}(3)}$ is the projection $\mathrm{Spin}^{\mathbb{C}}(3) \cong \mathrm{U}(2) \to \mathrm{SO}(3)$. Hence this class is included in $\mathcal{S}(Y)$. Thus, we can define $\nu(\mathfrak{s})$ as this class.

To prove the equivariance of ν, let $(\lambda_{\alpha\beta})$ be a U(1)-cocycle on Y (corresponding to some $\lambda \in \mathrm{H}^2(Y; \mathbb{Z})$). The $\mathrm{Spin}^{\mathbb{C}}$ structure $\lambda \cdot \mathfrak{s}$ is represented by the cocycle $(\widetilde{g}_{\alpha\beta}\,\lambda_{\alpha\beta})$. In $\mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)};\mathbb{Z})$ this corresponds to adding the pullback of λ to $\nu(\mathfrak{s})$ (the product is converted into a sum because there is an exponential map in the exact sequence of Lemma 1.41). Thus, ν is $\mathrm{H}^2(Y;\mathbb{Z})$-equivariant.

The bijectivity of ν follows from the $\mathrm{H}^2(Y;\mathbb{Z})$equivariance. □

3.2.2 Equivalence between Spin$^{\mathbb{C}}$ structures and Euler structures

The following theorem proves that in fact Spin$^{\mathbb{C}}$ structures and Euler structures in the case of 3-manifold are the same.

Theorem 3.19. *Let Y be a closed oriented connected 3-manifold. Then, there is a bijection* $\mathrm{nvect}(Y) \xrightarrow{\sim} \mathcal{S}(Y)$ *that is equivariant for the action of* $\mathrm{H}_1(Y;\mathbb{Z}) \cong \mathrm{H}^2(Y;\mathbb{Z})$.

Lemma 3.20. *Let $p : \mathrm{SO}(3) \to S^2$ be the spherical tangent bundle assigning to each triple (v_1, v_2, v_3) the first vector v_2.*

Then the pullback homomorphism $p^* : \mathrm{H}^2(S^2;\mathbb{Z}) \to \mathrm{H}^2(\mathrm{SO}(3);\mathbb{Z})$ sends any generator of $\mathrm{H}^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ to the non-zero element in $\mathrm{H}^2(\mathrm{SO}(3);\mathbb{Z}) \cong \mathbb{Z}_2$.

Proof. First, we have to prove that each fiber $p^{-1}(x)$ represents the non-zero element in $\mathrm{H}_1(\mathrm{SO}(3);\mathbb{Z})$. By homogeneity of the fiber bundle, we may assume that the fiber is $p^{-1}(e_1)$, e_1 being the first vector of the canonical basis of $\mathbb{R}^3$.

$\mathrm{SO}(3)$ may be seen as a 3-ball of radius π whose boundary is collapsed by the action of the antipodal map (i.e. as the projective space $\mathbb{RP}^3$) as follows. Each element v of the 3-ball is a vector in $\mathbb{R}^3$, to which we may associate the rotation on the axis containing v of angle $\|v\|$ in the direction defined by v through the 'right hand' rule (cf. Figure 3.6).

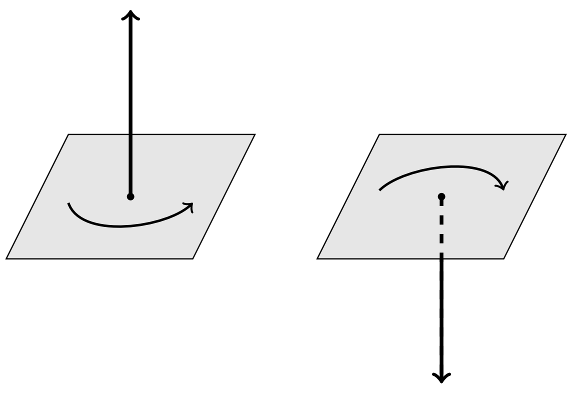


Figure 3.6: The figure shows the direction of the rotation induced on $\mathbb{R}^3$ by a certain vector v.

A vector v such that $\|v\| = \pi$ and its opposite $-v$ define the same rotation (since rotating by π is the same as rotating by $-\pi$), so the group of 3-dimensional rotations $\mathrm{SO}(3)$ is the quotient of the 3-ball by the action of the antipodal map on the boundary.

The fiber $p^{-1}(e_1)$ of the bundle $p : \mathrm{SO}(3) \to S^2$ is the projection on the quotient of the intersection between the 3-ball and the coordinate axis $\langle e_1 \rangle$ (cf. Figure 3.7).

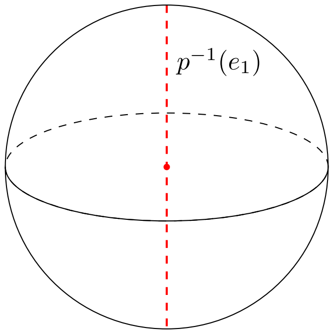


Figure 3.7: If $\mathrm{SO}(3)$ is represented as the quotient of the 3-ball in the picture, then the image of the dashed red line in the quotient represents the fiber $p^{-1}(e_1)$ (assuming that e_1 is vertically directed).

Hence, $p^{-1}(e_1)$ it represents the non-trivial element in $\mathrm{H}_1(\mathrm{SO}(3); \mathbb{Z})$.

Now we can prove the statement of the lemma. Consider a generator ω of $\mathrm{H}^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$. Its Poincaré dual is represented by any point $x \in S^2$. Then the Poincaré dual of $p^*(\omega)$ is represented by $p^{-1}(x)$ (cf. [BT82, §6, Poincaré Duality and the Thom Class, pag. 65-69]). As we just saw, the class $[p^{-1}(x)]$ in $\mathrm{H}_1(\mathrm{SO}(3); \mathbb{Z})$ is the non-zero element. Since Poincaré duality is an isomorphism, $p^*(\omega)$ has to be non-zero too. □

Proof of Theorem 3.19. Let $\eta : \mathcal{P}_{\mathrm{SO}(3)} \to Y$ represent the bundle of orthonormal frames, and let $\zeta : SY \to Y$ represent the bundle of normal tangent vectors. Let $p : \mathcal{P}_{\mathrm{SO}(3)} \to SY$ denote the morphism of bundles assigning to a triple of tangent vectors (v_1, v_2, v_3) the first vector v_1.

For each $y \in Y$, let $i_y : S^2 \to SY$ and $j_y : \mathrm{SO}(3) \to \mathcal{P}_{\mathrm{SO}(3)}$ the inclusions of the fibers over y. The commutativity of Diagram 3.5, together with Lemma 3.20, implies that the pullback homomorphism

$$p^* : \mathrm{H}^2(SY; \mathbb{Z}) \to \mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z})$$

maps $\mathrm{nvect}(Y) \subseteq \mathrm{H}^2(SY; \mathbb{Z})$ to $\mathcal{S}(Y) \subseteq \mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)}; \mathbb{Z})$ (because p^* sends any class that restricts to the generator of $\mathrm{H}^2(S_yY; \mathbb{Z})$ on each fiber $\zeta^{-1}(y)$ to a class that restricts to the non-trivial element in $\mathrm{H}^2(\eta^{-1}(y); \mathbb{Z}) \cong \mathbb{Z}_2$).

Moreover, p^* is equivariant for the action of $\mathrm{H}^2(Y; \mathbb{Z})$, because in both cases the action is by pullback and addition. The $\mathrm{H}^2(Y; \mathbb{Z})$-equivariance yields the bijectivity of the map p^* from $\mathrm{nvect}(Y)$ to $\mathcal{S}(Y)$. □

$$
\begin{tikzcd}
\mathrm{H}^2(SY;\mathbb{Z}) \arrow[r, "p^*"] \arrow[d, "i_y^*"] & \mathrm{H}^2(\mathcal{P}_{\mathrm{SO}(3)};\mathbb{Z}) \arrow[d, "j_y^*"] \\
\mathrm{H}^2(S^2;\mathbb{Z}) \arrow[r, "p^*"] & \mathrm{H}^2(\mathrm{SO}(3);\mathbb{Z})
\end{tikzcd}
$$

Diagram 3.5: The map $p : \mathcal{P}_{\mathrm{SO}(3)} \to S^2$ restricts to each fiber, so the above diagram commutes.

Thanks to Theorem 3.19, we will speak indifferently of $\mathrm{Spin}^{\mathbb{C}}$ structures and Euler structures when we deal with 3-dimensional closed connected oriented manifolds.

3.3 Turaev torsion

By Turaev torsion we mean the maximal abelian torsion defined by V. G. Turaev in [Tur02]. Other kinds of torsion will be necessary to define the maximal abelian torsion.

3.3.1 Torsion of a chain complex

Definition 3.21. Let V be a vector space over a field $\mathbb{K}$, and let a and a' be two ordered bases of V. $[{}^{a}/_{a'}]$ denotes the determinant of a change of basis matrix from the basis a' to the basis a:

$$[{}^{a}/_{a'}] = \det\Big(M_a^{a'}(\mathrm{id})\Big).$$

Let $C = (C_m \xrightarrow{\partial_m} \dots \xrightarrow{\partial_1} C_0)$ be a finite chain complex over a field $\mathbb{K}$, and for each $i = 0, \dots, m$ let c_i be a given basis of C_i and h_i be a given basis of $\mathrm{H}_i(C)$.

For each $i = 1, \dots, m$ choose a sequence of vectors b_i in C_i so that $\partial_i(b_i)$ is a basis of $\operatorname{Im}\partial_i$ and a lift $\widetilde{h}_i$ of the basis h_i to a sequence of vectors in C_i. Then the juxtaposition of the sequences of vectors $(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i$ yields a basis of C_i: indeed $(\partial_{i+1}b_{i+1})\widetilde{h}_i$ constitutes a basis of $\ker\partial_i$, whereas b_i is the preimage of a basis of $\operatorname{Im}\partial_i$.

Definition 3.22. The **torsion of a chain complex** C with respect to the bases c_i and h_i is defined as the element in $\mathbb{K}\setminus\{0\}$

$$\tau(C) = (-1)^{\varepsilon(C)} \prod_{i=0}^{m} \Big[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{c_i}\Big]^{(-1)^{i+1}}, \tag{3.13}$$

where

$$\varepsilon(C) = \sum_{i=0}^{m} \left(\sum_{r=0}^{i} \dim C_r \right) \left(\sum_{r=0}^{i} \dim \mathrm{H}_r(C) \right).$$

Remark. The definition of $\tau(C)$ does not depend on the lifts $\widetilde{h}_i$ of h_i nor on the choice of sequences of vectors b_i.

In fact, let $\widetilde{h}_i$ and $\widetilde{h}'_i$ be two lifts of h_i. Then the differences between the vectors in $\widetilde{h}_i$ and the vectors in $\widetilde{h}'_i$ are linear combinations of the vectors belonging to the basis $(\partial_{i+1}b_{i+1})$ of $\operatorname{Im}\partial_{i+1}$, hence

$$\left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{(\partial_{i+1}b_{i+1})\widetilde{h}'_i b_i} \right] = 1.$$

Since

$$\begin{aligned} \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{c_i} \right] &= \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{(\partial_{i+1}b_{i+1})\widetilde{h}'_i b_i} \right] \cdot \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}'_i b_i}/_{c_i} \right] \\ &= 1 \cdot \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}'_i b_i}/_{c_i} \right], \end{aligned}$$

$\tau(C)$ does not depend on the lift $\widetilde{h}_i$.

If b_i and b'_i are two sequences such that $\partial_i b_i = \partial_i b'_i$, the only factor in Equation (3.13) that may change substituting b_i with b'_i is

$$\left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{c_i} \right]^{(-1)^{i+1}}.$$

However, since the vectors in b_i and the vectors in b'_i differ by elements in $\ker(\partial_i)$ (which are linear combinations of vectors in $(\partial_{i+1}b_{i+1})\widetilde{h}_i$), the following equation holds:

$$\begin{aligned} \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{c_i} \right] &= \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b_i}/_{(\partial_{i+1}b_{i+1})\widetilde{h}_i b'_i} \right] \cdot \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b'_i}/_{c_i} \right] \\ &= 1 \cdot \left[{}^{(\partial_{i+1}b_{i+1})\widetilde{h}_i b'_i}/_{c_i} \right]. \end{aligned}$$

Thus, $\tau(C)$ does not change if we choose different b'_i such that $\partial_i b_i = \partial_i b'_i$.

Finally, let b_i and b'_i be sequences of vectors such that $\partial_i b_i \neq \partial_i b'_i$. Let $M^{\partial_i b_i}_{\partial_i b'_i}(\mathrm{id})$ be the matrix that expresses the vectors of the basis $\partial_i b'_i$ with respect to the basis ∂_i. Let b''_i be the sequence of vectors such that the matrix $M^{\partial_i b_i}_{\partial_i b'_i}(\mathrm{id})$ carries b_i to b''_i. Thus,

$$\left[{}^{b''_i}/_{b_i} \right] = \left[{}^{\partial_i b''_i}/_{\partial_i b_i} \right]. \tag{3.14}$$

Moreover, $\partial_i b''_i = \partial_i b'_i$ because they are both obtained from $\partial_i b_i$ by applying the matrix $M^{\partial_i b_i}_{\partial_i b'_i}(\mathrm{id})$. By the result of the previous paragraph, the torsion does not change if we replace b'_i with b''_i. Thus, we just have to check what happens if we replace b_i with b_i''. The factors that change in the expression of $\tau(C)$ (cf. Equation (3.13)) when we replace b_i with b_i'' are

$$\left[(\partial_i b_i)\widetilde{h}_{i-1}b_{i-1}/_{c_{i-1}}\right]^{(-1)^i} \quad \text{and} \quad \left[(\partial_{i+1} b_{i+1})\widetilde{h}_i b_i/_{c_i}\right]^{(-1)^{i+1}}.$$

The former is multiplied by $\left[\partial_i b_i''/_{\partial_i b_i}\right]^{(-1)^i}$, whereas the latter is multiplied by $\left[b_i''/_{b_i}\right]^{(-1)^{i+1}}$. Equation (3.14) implies that $\tau(C)$ does not change.

Remark. If C is an *acyclic complex*, the basis h_i is empty, so the torsion depends only on the basis c_i and it is expressed by the formula

$$\tau(C) = \prod_{i=0}^{m} \left[(\partial_{i+1} b_{i+1})b_i/_{c_i}\right]^{(-1)^{i+1}} \in \mathbb{K} \setminus \{0\}. \tag{3.15}$$

3.3.2 Reidemeister-Franz torsion

Let A be a finite connected CW complex, and let $\varphi : \mathbb{Z}\left[\mathrm{H}_1(A;\mathbb{Z})\right] \to \mathbb{K}$ be a ring homomorphism. Let $\widetilde{A} \to A$ denote the maximal abelian cover of A (i.e. the cover associated to the commutator subgroup of $\pi_1(A)$). Denote by $H = \mathrm{H}_1(A;\mathbb{Z}) \cong \mathrm{Aut}_A(\widetilde{A})$.

Let $\mathrm{C}^{\varphi}_*(A)$ be the chain complex

$$\mathrm{C}^{\varphi}_*(A) = \mathbb{K} \otimes_{\mathbb{Z}[H]} \mathrm{C}_*(\widetilde{A};\mathbb{Z}),$$

where the boundary map is $\mathrm{id} \otimes \partial_{\widetilde{A}}$.

We would like to define the Reidemeister-Franz torsion as the torsion of the chain complex $\mathrm{C}^{\varphi}_*(A)$, but we would need a basis of $\mathrm{C}^{\varphi}_*(A)$ and a basis of $\mathrm{H}_*(\mathrm{C}^{\varphi}_*(A))$ to define it. We will focus on the cases in which $\mathrm{H}_*(\mathrm{C}^{\varphi}_*(A))$ vanishes, so we only need to specify an ordered basis of $\mathrm{C}^{\varphi}_*(A)$. Such a basis will be given by the choice of a fundamental family of the covering $\widetilde{A} \to A$.

Definition 3.23. Let $\pi : B \to A$ be a covering map of CW complexes. A **fundamental family** e is a collection of cells in B such that for each cell of A a there exists a unique cell b in the collection such that $\pi(b) = a$.

A fundamental family e of the covering map $\widetilde{A} \to A$ induces a basis of the complex $\mathrm{C}^{\varphi}_*(A)$ for each φ (by projection). Note that if φ is the augmentation map, which sends each $h \in H$ to $1 \in \mathbb{K}$, the complex $\mathrm{C}^{\varphi}_*(A)$ is $\mathrm{C}_*(A;\mathbb{K})$, and the basis given by the fundamental family is simply the collection of all cells in A (with some orientation). Here is a first problem: the choice of a fundamental family does not specify the orientation of the cells.

Moreover, a fundamental family e determines a basis of $\mathrm{C}^{\varphi}_*(A)$, but, in order to define the torsion, we would need an ordered basis. The choice of an order of the basis induced by e changes the torsion of the complex by multiplication of ± 1.

For these reasons, a fundamental family is not sufficient to determine the torsion, but also a homological orientation of A is needed.

Definition 3.24. Let A be a topological space. A **homological orientation** of A is an orientation of the vector space $\mathrm{H}_*(A;\mathbb{R})$.

Remark 3.25. If W is an odd-dimensional closed connected manifold, then there is a canonical homological orientation, given by Poincaré duality. Choose an ordered basis h_i of $\mathrm{H}_i(W;\mathbb{R})$ for each $0 \leq i \leq \frac{\dim W-1}{2}$, and let $h_{\dim W-2i}$ be the basis of $\mathrm{H}_{\dim W-2i}(W;\mathbb{R})$ dual to h_i. Then, the juxtaposition of these basis $h_0 h_1 \dots h_{\dim W}$ yields a well-defined homological orientation of W.

Let ω be a homological orientation for A, and suppose that an ordered basis c of $\mathrm{C}_*(A;\mathbb{R})$ is given. Then the sign of the torsion $\tau(\mathrm{C}_*(A;\mathbb{R}))$ does not depend on the choice of a basis of $\mathrm{H}_*(A;\mathbb{R})$, provided that such a basis gives the orientation ω to $\mathrm{H}_*(A;\mathbb{R})$. Let $\tau_0(A,\omega,c) \in \{\pm 1\}$ denote this sign.

Definition 3.26. Let A be a finite connected CW complex with a homological orientation ω and let $\varphi : \mathbb{Z}[H] \to \mathbb{K}$ be a ring homomorphism. Let e be a fundamental family for the covering map $\widetilde{A} \to A$.

Choose an order and an orientation of the cells of e. Then bases c and c' are induced by e respectively on $\mathrm{C}_*(A;\mathbb{R})$ and $\mathrm{C}^{\varphi}_*(A)$.

The **Reidemeister-Franz φ-torsion** of (A,ω) with respect to the fundamental family e is

$$\tau^{\varphi}(A,\omega,e) = \begin{cases} \tau_0(A,\omega,c)\cdot\tau(\mathrm{C}^{\varphi}_*(A)) & \text{if } \mathrm{C}^{\varphi}_*(A) \text{ is acyclic} \\ 0 & \text{if } \mathrm{H}_*(\mathrm{C}^{\varphi}_*(A)) \neq 0 \end{cases}$$

where $\mathrm{C}^{\varphi}_*(A)$ is endowed with the ordered basis c' and $\mathrm{H}_*(\mathrm{C}^{\varphi}_*(A);\mathbb{Z})$ is endowed with the empty basis.

Remark. If the orientation of a cell of e is reversed or if the order of the cells of e is changed by switching two cells, then both $\tau_0(A,\omega,c)$ and $\tau(\mathrm{C}^{\varphi}_*(A))$ are multiplied by -1, so $\tau^{\varphi}(A,\omega,e)$ depends only on the family e and not on a choice of the order and the orientations of the cells belonging to it.

3.3.3 Torsion of Euler structures

For a finite connected CW complex A with a homological orientation ω, a fundamental family e and a ring homomorphism $\varphi : \mathbb{Z}[\mathrm{H}_1(A;\mathbb{Z})] \to \mathbb{K}$ (where $\mathbb{K}$ is a field) the Reidemeister-Franz φ-torsion $\tau^{\varphi}(A,\omega,e) \in \mathbb{K}$ has been defined (cf. Section 3.3.2).

Suppose now that $\chi(A) = 0$. We will prove in this section that any fundamental family e defines an Euler structure on A and that the torsion $\tau^{\varphi}(A,\omega,e)$ depends not on e but only on the associated Euler structure.

Let e be a fundamental family for the covering map $\widetilde{A} \to A$. An associated Euler chain can be constructed as follows. Choose a point $x \in \widetilde{A}$ and then choose arbitrary paths in $\widetilde{A}$ that link x to the centres of each cell of the fundamental family (oriented towards x if the cell is odd-dimensional and oriented out of x if the cell is even-dimensional). Projecting this 1-chain to A we get an Euler chain ϑ_e. The class $[\vartheta_e] \in \mathrm{Eul}(A)$ does not depend on the choice of the paths: indeed the change of the path from the centre $x_{\widetilde{a}}$ of a cell $\widetilde{a}$ to x in $\widetilde{A}$ adds to ϑ_e the cycle obtained by joining the two paths (one of them with reversed orientation) and projecting this loop to A. Since $\widetilde{A}$ is the maximal abelian cover of A, any loop in $\widetilde{A}$ projects to a homologically trivial loop in A. Hence $[\vartheta_e]$ does not depend on the choice of the paths. Analogously the choice of the 'base point' $x \in \widetilde{A}$ does not affect the class $[\vartheta_e]$. Hence there exists a map

$$[\vartheta.] : \mathrm{FF}(\widetilde{A} \to A) \longrightarrow \mathrm{Eul}(A),$$

where $\mathrm{FF}(\widetilde{A} \to A)$ represents the set of fundamental families of the cover $\widetilde{A} \to A$, assigning to a fundamental family e the Euler structure $[\vartheta_e]$.

For each $h \in \mathrm{H}_1(A;\mathbb{Z}) = H$ and for each cell $\widetilde{a} \in \widetilde{A}$ it is possible to define the translation of $\widetilde{a}$ by h (because there is an identification $H \cong \mathrm{Aut}_A(\widetilde{A})$). If e is a fundamental family and e' is the fundamental family obtained by translating a cell $\widetilde{a} \in e$ by h, then

$$[\vartheta_{e'}] = \left((-1)^{\dim \widetilde{a}} h\right) \cdot [\vartheta_e]. \tag{3.16}$$

By Equation (3.15) the torsions satisfy the following relation

$$\begin{aligned} \tau^{\varphi}(A,\omega,e') &= \varphi(h)^{(-1)^{\dim \widetilde{a}}} \cdot \tau^{\varphi}(A,\omega,e) \\ &= \varphi((-1)^{\dim \widetilde{a}} h) \cdot \tau^{\varphi}(A,\omega,e) \end{aligned} \tag{3.17}$$

(recall that $\varphi : \mathbb{Z}[H] \to \mathbb{K}$ is a ring homomorphism, where in $\mathbb{Z}[H]$ the multiplication is the addition in H).

Equation (3.16) incidentally proves that $[\vartheta.]$ is a surjective map. Moreover, if $[\vartheta_e]$ and $[\vartheta_{e'}]$ are Euler structures induced by fundamental families e and e', then e' can be obtained from e by translating each cell $\widetilde{a}$ by a certain $h_{\widetilde{a}} \in H$. Equations (3.16) and (3.17) prove that the torsion $\tau^{\varphi}(A,\omega,e)$ does not depend on e but just on $[\vartheta_e]$ and that, if we define $h = \sum_{\widetilde{a} \in e} (-1)^{\dim \widetilde{a}} h_{\widetilde{a}}$, then

$$\tau^{\varphi}(A,\omega,e') = \varphi(h) \cdot \tau^{\varphi}(A,\omega,e). \tag{3.18}$$

Thus, the following definition can now be given.

Definition 3.27. Let A be a finite connected CW complex associated to a manifold W, with a homological orientation ω, and let $\varphi : \mathbb{Z}[H] \to \mathbb{K}$ be a ring homomorphism. Suppose that $\chi(A) = 0$ and let $\mathfrak{s} \in \mathrm{Eul}(A) \cong \mathrm{Spin}^{\mathbb{C}}(W)$ (cf. Theorems 3.9, 3.15 and 3.19 and Lemma 3.18).

The **φ-torsion** of A with respect to ω and the Euler structure $\mathfrak{s}$ is

$$\tau^{\varphi}(A,\omega,\mathfrak{s}) = \tau^{\varphi}(A,\omega,e) \in \mathbb{K},$$

where e is any fundamental family of the maximal abelian cover $\widetilde{A} \to A$ such that $[\vartheta_e]$ corresponds to $\mathfrak{s}$ (cf. Definition 3.26).

Remark. Equation (3.18) implies that for each $\mathfrak{s} \in \mathrm{Eul}(A)$ and for each $h \in H = \mathrm{H}_1(A;\mathbb{Z})$ the following relation holds:

$$\tau^{\varphi}(A,\omega,h\cdot\mathfrak{s}) = \varphi(h)\cdot\tau^{\varphi}(A,\omega,\mathfrak{s}). \tag{3.19}$$

3.3.4 Maximal abelian torsion

Algebraic preliminaries

Throughout this subsection H will be a finite abelian group. Let $\mathrm{Char}(H)$ denote the group of the characters, i.e. the set of group homomorphisms $\sigma : H \to \mathbb{C}^*$ with the product

$$(\sigma\cdot\rho)(h) = \sigma(h)\cdot\rho(h).$$

Two characters σ and ρ are equivalent (or conjugated) if there exists a field automorphism $\psi \in \mathrm{Aut}(\mathrm{Im}\,\sigma)$ such that

$$\rho = \psi \circ \sigma.$$

Let $\sigma_1,\ldots,\sigma_k$ characters such that there exists a unique representative σ_i for each equivalence class. Let C_{σ_i} denote the cyclotomic field $\mathbb{Q}\left[\mathrm{Im}(\sigma_i)\right] = \mathbb{Q}\left[\zeta_{n_i}\right] \subseteq \mathbb{C}^*$.

Consider the map

$$(\sigma_1,\ldots,\sigma_k) : H \to \bigoplus_{i=1}^{k} C_{\sigma_i}$$

and extend it by linearity to a map

$$\varphi : \mathbb{Q}\left[H\right] \to \bigoplus_{i=1}^{k} C_{\sigma_i}.$$

Then the following proposition states that φ is an isomorphism.

Proposition 3.28. *If H is a finite abelian group and $\sigma_1,\ldots,\sigma_k$ are representatives of the equivalence classes of characters $H \to \mathbb{C}^*$, the map*

$$\varphi : \mathbb{Q}\left[H\right] \to \bigoplus_{i=1}^{k} C_{\sigma_i}$$

is an isomorphism.

Proof. Since C_{σ_i} is cyclotomic, it is also a Galois field, hence

$$|\,\mathrm{Aut}_{\mathbb{Q}}\, C_{\sigma_i}| = \dim_{\mathbb{Q}} C_{\sigma_i}.$$

Thus, the number of characters conjugated to σ_i is $\dim_{\mathbb{Q}} C_{\sigma_i}$. Therefore

$$|\operatorname{Char}(H)| = \sum_{i=1}^{k} \dim_{\mathbb{Q}} C_{\sigma_i} = \dim_{\mathbb{Q}} \left(\bigoplus_{i=1}^{k} C_{\sigma_i} \right).$$

Since H is abelian, $|\operatorname{Char}(H)| = |H|$ (cf. [Ser77, Ex. 3.3]), hence

$$\dim_{\mathbb{Q}} \mathbb{Q}\,[H] = |H| = |\operatorname{Char}(H)| = \dim_{\mathbb{Q}} \left(\bigoplus_{i=1}^{k} C_{\sigma_i} \right).$$

Therefore, the dimensions of the domain and of the codomain of φ are the same, so it is sufficient to prove the injectivity of φ.

If $y \in \mathbb{Q}\,[H]$ is such that $\varphi(y) = 0$, then $\sigma_i(y) = 0$ for each i, hence (by conjugating the characters σ_i) $\sigma(y) = 0$ for each character $\sigma \in \operatorname{Char}(H)$. Hence it is sufficient to prove that if $y \in \mathbb{Q}\,[H]$ is a zero for all characters, then it is 0. We will prove this statement also for all $y \in \mathbb{C}\,[H]$.

By the structure theorem H is the direct sum of a finite number of cyclic groups:

$$H = \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_s}.$$

Let $x_1, \ldots, x_s$ be generators of the cyclic groups $\mathbb{Z}_{a_1}, \ldots, \mathbb{Z}_{a_s}$. For each multi-index $\mathcal{B} = (b_1, \ldots, b_s) \in \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_s}$, the unique group homomorphism $\sigma_{\mathcal{B}} : H \to \mathbb{C}^*$ such that for each $j = 1, \ldots, s$

$$\sigma_{\mathcal{B}} : x_j \longmapsto \zeta_{a_j}^{b_j}$$

is a character. Different multi-indices yield different characters. As the multi-indices are as many as $|H|$, the characters $\sigma_{\mathcal{B}}$ constitute the whole set $\operatorname{Char}(H)$.

Now for each multi-index $\mathcal{B}$ consider the linear extension of the character $\sigma_{\mathcal{B}}$ to

$$\overline{\sigma_{\mathcal{B}}} : \mathbb{C}\,[H] \longrightarrow \mathbb{C}.$$

Endow the set of the multi-indices with the lexicographical order, and gather all the maps $\overline{\sigma_{\mathcal{B}}}$ in a single map

$$\xi_H : \mathbb{C}\,[H] \longrightarrow \mathbb{C}^{|H|}$$

that associates to each element $y \in \mathbb{C}\,[H]$ the $|H|$-tuple of complex numbers $\overline{\sigma_{\mathcal{B}}}(y)$.

The statement to be proved is that ξ_H is injective. Since it is a linear map of $|H|$-dimensional $\mathbb{C}$-vector spaces, it is sufficient to calculate its determinant with respect to some bases and check that it does not vanish.

Choose on $\mathbb{C}\,[H]$ the basis

$$\left\{ x^{\mathcal{C}} = x_1^{c_1} \cdots x_s^{c_s} \,\middle|\, \mathcal{C} = (c_1, \ldots, c_s) \in \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_s} \text{ multi-index} \right\}$$

with the lexicographical order and choose on $\mathbb{C}^{|H|}$ the standard basis

$$\{e_{\mathcal{C}} = e_{c_1,\dots,c_s} \,|\, \mathcal{C} = (c_1, \dots, c_s) \in \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_s} \text{ multi-index}\}$$

again with the lexicographical order.

For each

$$y = \sum_{\mathcal{C} \in \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_s}} \alpha_{\mathcal{C}}\, x^{\mathcal{C}} \in \mathbb{C}\,[H]$$

and for each multi-index $\mathcal{B}$,

$$\overline{\sigma_{\mathcal{B}}}(y) = \alpha_{\mathcal{C}}\, \zeta_{a_1}^{b_1 c_1} \cdots \zeta_{a_s}^{b_s c_s}.$$

Hence, the matrix M that represents ξ_H with respect to these bases is

$$\left(\zeta_{a_1}^{b_1 c_1} \cdots \zeta_{a_s}^{b_s c_s}\right)_{\mathcal{B},\mathcal{C}}.$$

M is a matrix that is very similar to a Vandermonde matrix. If N is the matrix that represents $\xi_{\widetilde{H}}$, where $\widetilde{H}$ is the subset of H defined by

$$\widetilde{H} = \{0\} \times \mathbb{Z}_{a_2} \times \cdots \times \mathbb{Z}_{a_s} \subseteq \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_s} = H,$$

then the matrix M is

$$\begin{pmatrix} N & N & N & \cdots & N \\ N & \zeta_{a_1} N & \zeta_{a_1}^2 N & \cdots & \zeta_{a_1}^{a_1-1} N \\ N & \zeta_{a_1}^2 N & \zeta_{a_1}^4 N & \cdots & \zeta_{a_1}^{a_1-2} N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N & \zeta_{a_1}^{a_1-1} N & \zeta_{a_1}^{a_1-2} N & \cdots & \zeta_{a_1} N \end{pmatrix}.$$

The determinant of M can be calculated by using the same tricks of the calculation of the determinant of a Vandermonde matrix. The result is

$$\det M = (\det N)^{a_1} \cdot \left[\prod_{\substack{d>e \\ d,e=0,\dots,a_1-1}} \left(\zeta_{a_1}^d - \zeta_{a_1}^e\right) \right]^{\frac{|H|}{a_1}}.$$

By induction on s the above formula implies that

$$\det M = \prod_{j=1}^{s} \left[\prod_{\substack{d_j>e_j \\ d_j,e_j=0,\dots,a_j-1}} \left(\zeta_{a_j}^{d_j} - \zeta_{a_j}^{e_j}\right) \right]^{\frac{|H|}{a_j}}.$$

Thus, $\det M \neq 0$, hence the map ξ_H is injective. The proposition is proved. $\square$

Finally, the splitting

$$\mathbb{Q}[H] \cong \bigoplus_{i=1}^{k} C_{\sigma_i}$$

is unique up to permutation of the summands thanks to Lemma 3.29.

Lemma 3.29. *If a ring R splits as the sum of finite fields $\mathbb{K}_i$, then the the fields are determined.*

Proof. The fields $\mathbb{K}_i$ are in bijective correspondence with the maximal ideals of R. □

The definition of the maximal abelian torsion

Definition 3.30. Let A be a finite connected CW complex associated to a manifold W, with a homological orientation ω. Suppose that $\chi(A) = 0$ and let $\mathfrak{s} \in \mathrm{Eul}(A) \cong \mathrm{Spin}^{\mathbb{C}}(W)$.

By Proposition 3.28 the ring $\mathbb{Q}[H] = \mathbb{Q}[\mathrm{H}_1(A;\mathbb{Z})]$ splits as the sum of a finite number of fields $\mathbb{K}_i$:

$$\pi : \mathbb{Q}[H] \xrightarrow{\sim} \bigoplus_{i=1}^{k} \mathbb{K}_i.$$

For each i, let $\pi_i : \mathbb{Q}[H] \to \mathbb{K}_i$ denote the projection to the quotient, and let φ_i be the composition of the inclusion $\mathbb{Z}[H] \to \mathbb{Q}[H]$ and π_i.

The **maximal abelian torsion** of A with respect to ω and the Euler structure $\mathfrak{s}$ is

$$\tau(A, \omega, \mathfrak{s}) = \pi^{-1}\left(\sum_{i=1}^{k} \tau^{\varphi_i}(A, \omega, \mathfrak{s})\right) \in \mathbb{Q}[H]. \tag{3.20}$$

Remark. $\tau(A, \omega, \mathfrak{s})$ is a well defined element of $\mathbb{Q}[H]$ thanks to Lemma 3.29.

Remark. As a consequence of Equation (3.19), the following identity on the maximal abelian torsion holds:

$$\tau(A, \omega, h \cdot \mathfrak{s}) = h \cdot \tau(A, \omega, \mathfrak{s}). \tag{3.21}$$

Remark. Sometimes it will be necessary to focus on the rational coefficients of the torsion $\tau(A, \omega, \mathfrak{s}) \in \mathbb{Q}[H]$. They are denoted by $\tau(A, \omega, \mathfrak{s}, h)$:

$$\tau(A, \omega, \mathfrak{s}) = \sum_{h \in H} \tau(A, \omega, \mathfrak{s}, h) \cdot h \in \mathbb{Q}[H]. \tag{3.22}$$

Note that Equations (3.21) and (3.22) imply that

$$\tau(A, \omega, h \cdot \mathfrak{s}, g) = \tau(A, \omega, \mathfrak{s}, h^{-1} \cdot g). \tag{3.23}$$

Remark. If A is an odd-dimensional closed connected manifold, we will assume that it is endowed with this standard homological orientation ω of Remark 3.25. Hence we will omit the homological orientation ω each time A is odd-dimensional, and we will denote $\tau(A, \omega, \mathfrak{s}, h)$ by $\tau(A, \mathfrak{s}, h)$.

3.3.5 Calculating the torsion

Let Y be a 3-dimensional closed connected oriented manifold. Then there exists a handle decomposition in one 0-handle, m 1-handles, m 2-handles and one 3-handle or, analogously, there exists a cellular structure A of Y with one 0-cell, m 1-cells, m 2-cells and one 3-cell (just taking the cores of the handles). The 0-cell and the 3-cell are oriented in a natural way. Choose an orientation also for the 1-cells and the 2-cells. Denote the 1-cells by $e_1, \dots, e_m$ and the 2-cells by $f_1, \dots, f_m$. The chain complex associated to A is the following:

$$\begin{tikzcd}
C_3(A;\mathbb{Z}) \arrow[r, "\partial_3"] \arrow[d, "\wr"] & C_2(A;\mathbb{Z}) \arrow[r, "\partial_2"] \arrow[d, "\wr"] & C_1(A;\mathbb{Z}) \arrow[r, "\partial_1"] \arrow[d, "\wr"] & C_0(A;\mathbb{Z}) \arrow[d, "\wr"] \\
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}^m \arrow[r, "M"] & \mathbb{Z}^m \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}$$

The map ∂_1 is the zero map because each 1-cell e_i is attached to the unique 0-cell at the edges, whereas the map ∂_3 vanishes because the unique 3-cell must be a cycle (since $\mathrm{H}_3(A;\mathbb{Z}) \cong \mathbb{Z}$). The map ∂_2 is represented by a matrix M, which is a presentation matrix for $\mathrm{H}_1(A;\mathbb{Z})$, with respect to the basis $f_1, \dots, f_m$ and $e_1, \dots, e_m$. Note that such a matrix is the abelianization of a presentation matrix for $\pi_1(A)$, in which the generators of $\pi_1(A)$ are given by the 1-cells $e_1, \dots, e_m$, and the relations are obtained by imposing that the boundary of each 2-cell is 0.

Let $\widetilde{A} \to A$ be the maximal abelian cover of A, and denote by H the group $\mathrm{H}_1(A;\mathbb{Z})$. The choice of a fundamental family e of the cover gives a structure of $\mathbb{Z}[H]$-module to the complex of cellular chains of $\widetilde{A}$: each cell in the fundamental family is given the coefficient $e_H \in H$, and the other ones are obtained by translating the cells in e by a certain $h \in H \cong \mathrm{Aut}_A(\widetilde{A})$. Hence the diagram

$$\begin{tikzcd}
C_3(\widetilde{A};\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & C_2(\widetilde{A};\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & C_1(\widetilde{A};\mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & C_0(\widetilde{A};\mathbb{Z}) \arrow[d, "\wr"] \\
\mathbb{Z}[H] \arrow[r, "\widetilde{\partial}_3"] \arrow[d, "\varepsilon"] & (\mathbb{Z}[H])^m \arrow[r, "\widetilde{\partial}_2"] \arrow[d, "\varepsilon^m"] & (\mathbb{Z}[H])^m \arrow[r, "\widetilde{\partial}_1"] \arrow[d, "\varepsilon^m"] & \mathbb{Z}[H] \arrow[d, "\varepsilon"] \\
\mathbb{Z} \arrow[r, "0"] \arrow[d, "\wr"] & \mathbb{Z}^m \arrow[r, "M"] \arrow[d, "\wr"] & \mathbb{Z}^m \arrow[r, "0"] \arrow[d, "\wr"] & \mathbb{Z} \arrow[d, "\wr"] \\
C_3(A;\mathbb{Z}) \arrow[r] & C_2(A;\mathbb{Z}) \arrow[r] & C_1(A;\mathbb{Z}) \arrow[r] & C_0(A;\mathbb{Z})
\end{tikzcd}$$

commutes, where $\varepsilon : \mathbb{Z}[H] \to \mathbb{Z}$ represents the augmentation map. Let $\widetilde{e}_1, \dots, \widetilde{e}_m$ represent the 1-cells belonging to the fundamental family e, and $\widetilde{f}_1, \dots, \widetilde{f}_m$ represent the 2-cells belonging to e (so that $\widetilde{e}_i$ projects to e_i and $\widetilde{f}_i$ projects to f_i).

Consider the map $\widetilde{\partial}_1$. Each element of the basis $\widetilde{e}_i$ is a mapped to some $h_i - \overline{h_i}$ (here h_i and $\overline{h_i}$ represent the starting and the ending point of $\widetilde{e}_i$). Up to changing the fundamental family, we may assume that

$$\widetilde{\partial}_1(\widetilde{e}_i) = h_i - 1.$$

Note that the element h_i represents the homology class of the oriented 1-cell e_i, hence $h_1, \dots, h_m$ generate H (they are exactly the generators of the presentation matrix M).

Focus now on the map $\widetilde{\partial}_3$. The unique 3-cell of A bounds each 2-cell twice (once positively and once negatively). Thus, the boundary of the 3-cell in $\widetilde{A}$ that belongs to the fundamental family is

$$(g_1 - \overline{g_1}) \cdot \widetilde{f}_1 + \dots + (g_m - \overline{g_m}) \cdot \widetilde{f}_m$$

for some $g_i, \overline{g_i} \in H$, i.e. it is represented by the vector

$$\begin{pmatrix} g_1 - \overline{g_1} \\ \vdots \\ g_m - \overline{g_m} \end{pmatrix}.$$

Again, up to changing the 2-cells in the fundamental family, we may assume that $\overline{g_i} = 1$ for all i. Hence, the map $\widetilde{\partial}_3$ is represented by the vector

$$\begin{pmatrix} g_1 - 1 \\ \vdots \\ g_m - 1 \end{pmatrix}.$$

Note that g_i is the homology class of a loop in A that intersects the 2-cell f_i once (positively) and does not intersect any other cell. Such a loop is for instance the projection of a curve in the 3-cell of $\widetilde{A}$ that belongs to the fundamental family e, which starts from $\widetilde{f}_i$ and ends in $g_i \cdot \widetilde{f}_i$ (see Figure 3.8).

Finally, we should find the matrix that represents the map $\widetilde{\partial}_2$. For each i, the boundary $\partial(f_i)$ is a word in the 1-cells $e_1^{\pm 1}, \dots, e_m^{\pm 1}$.

Remark 3.31. Note that the relation b_i defining the boundary of f_i in terms of the cells $e_1^{\pm 1}, \dots, e_m^{\pm 1}$ is not determined: since the boundary $\partial(f_i)$ is not a based loop, b_i is determined up to conjugacy. Hence, a choice of the relations b_i (in the conjugacy class) is required.

Once the relations b_i are fixed, we can express the boundary of $\widetilde{f}_i$ in $\widetilde{A}$ in homology as the sum of lifts of the 1-cells (which generally are not $\widetilde{e}_1, \dots, \widetilde{e}_m$).

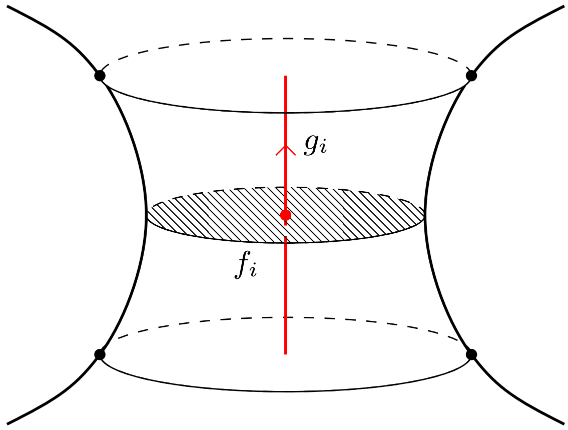


Figure 3.8: $g_i \in H$ is represented by a loop in A that intersects $\widetilde{f}_i$ once (positively) and does not intersect any other cell.

For instance, assume that

$$\partial(f_1) = e_1 e_2 e_1^{-1} e_2^{-1}, \tag{3.24}$$

and that the cells $\widetilde{e}_1$, $\widetilde{e}_2$ and $\widetilde{f}_1$ in $\widetilde{A}$ appear as in Figure 3.9. If h_1 and h_2 are the homology classes of e_1 and e_2, then

$$\begin{aligned}\widetilde{\partial}_2(\widetilde{f}_1) &= h_1^2 h_2 \widetilde{e}_1 + h_1^3 h_2 \widetilde{e}_2 - h_1^2 h_2^2 \widetilde{e}_1 - h_1^2 h_2 \widetilde{e}_2 \\ &= (h_1^2 h_2 - h_1^2 h_2^2)\widetilde{e}_1 + (h_1^3 h_2 - h_1^2 h_2)\widetilde{e}_2 \\ &= h_1^2 h_2 \left[(1 - h_2)\widetilde{e}_1 + (h_1 - 1)\widetilde{e}_2\right].\end{aligned}$$

The coefficients of the boundary of $\widetilde{f}_1$ are related with the Fox derivatives of the relation in Equation (3.24) (the definition of Fox derivative can be found in [Fox53]). They are actually the Fox derivatives of the relation given by Equation (3.24), up to the translation by $h_1^2 h_2$, which is the element of H that carries the 0-cell of the fundamental family to the base point of the boundary relation (in Figure 3.9 it is the element that carries the black point, which represents the 0-cell in the fundamental family, to the red one).

Generally speaking, the same holds for any 2-cell of the fundamental family. Consider the boundary $\widetilde{\partial}_2(\widetilde{f}_i)$ in homology. The coefficient of $\widetilde{e}_j$ in the expression of the boundary is the j-th Fox derivative of the relation b_i that expresses the boundary of f_i in A in terms of the 1-cells $e_1, \ldots, e_m$, multiplied by an element $l_i \in H$ that expresses the translation of the base point in $\widetilde{A}$, from the 0-cell in the fundamental family to the point in the boundary of $\widetilde{f}_i$ that corresponds to the 'starting point' of the relation b_i (the red point in Figure 3.9).

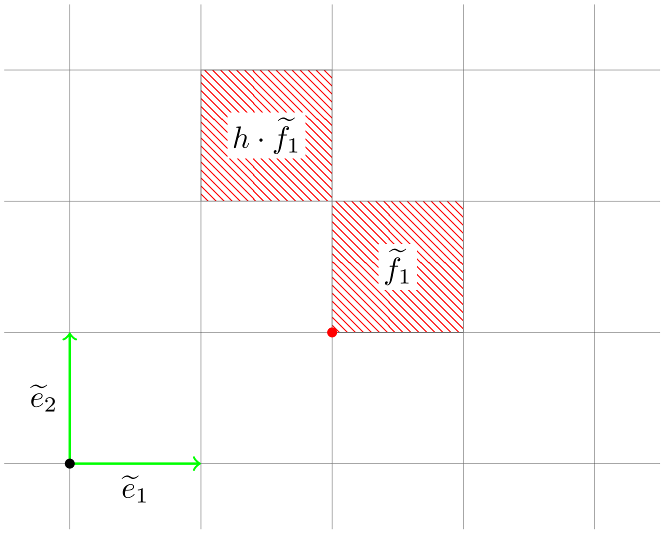


Figure 3.9: If h_1 and h_2 are the homology classes of the cells e_1 and e_2, then in the case in the picture we have $\partial(\widetilde{f}_1) = h_1^2 h_2 \left[(1-h_2)\widetilde{e}_1 + (h_1 - 1)\widetilde{e}_2\right]$ and $\partial(h \cdot \widetilde{f}_1) = h h_1^2 h_2 \left[(1-h_2)\widetilde{e}_1 + (h_1 - 1)\widetilde{e}_2\right]$ (where $h = h_1^{-1}h_2$).

Thus, the matrix representing $\widetilde{\partial}_2$ is

$$\widetilde{M} = \begin{pmatrix} l_1\partial_1(b_1) & \cdots & l_m\partial_1(b_m) \\ \vdots & \ddots & \vdots \\ l_1\partial_m(b_1) & \cdots & l_m\partial_m(b_m) \end{pmatrix}, \tag{3.25}$$

where ∂_i denotes the j-th Fox derivative.

Remark. If a 2-cell of the fundamental family $\widetilde{f}_i$ is translated by some $h \in H$, then the element l_i is multiplied by h. For instance, consider the cell $h \cdot \widetilde{f}_i$ in Figure 3.9. Its boundary is the boundary of the 2-cell $\widetilde{f}_i$ multiplied by h. Hence, the choice of the fundamental family affects the entries of $\widetilde{M}$ in the following way: if the 2-cell $\widetilde{f}_i$ is translated by h, the i-th column of the matrix is multiplied by h.

Now all the maps $\widetilde{\partial}_1$, $\widetilde{\partial}_2$ and $\widetilde{\partial}_3$ are expressed as matrices with respect to the basis given by the fundamental family, so we are able to calculate the torsion. Let $\varphi : \mathbb{Q}[H] \to \mathbb{K} \subseteq \mathbb{C}$ be a ring homomorphism, which endows $\mathbb{K}$ with a structure of $\mathbb{Z}[H]$-module. Consider the φ-twisted complex $\mathrm{C}_*^{\varphi}(A;\mathbb{Z}) = \mathrm{C}_*(\widetilde{A};\mathbb{Z}) \otimes_{\mathbb{Z}[H]} \mathbb{K}$. The fundamental family chosen above provides bases c_0, c_1, c_2, c_3 of $\mathrm{C}_i^{\varphi}(A;\mathbb{Z})$, and with respect to these bases the chain complex is

$$\mathbb{K} \xrightarrow{\begin{pmatrix}\varphi(g_1)-1\\ \vdots \\ \varphi(g_m)-1\end{pmatrix}} \mathbb{K}^m \xrightarrow{\qquad\varphi(\widetilde{M})\qquad} \mathbb{K}^m \xrightarrow{(\varphi(h_1)-1,\ldots,\varphi(h_m)-1)} \mathbb{K}$$

where $\widetilde{M}$ is the one in Equation (3.25).

Suppose now that $\varphi(g_r - 1) \neq 0$, $\varphi(h_s - 1) \neq 0$ and $\varphi(\Delta^{r,s}) \neq 0$, where $\Delta^{r,s}$ is the determinant of the minor of $\widetilde{M}$ obtained by deleting the r-th column and the s-th row. Then the maps $\mathbb{K} \to \mathbb{K}^m$ and $\mathbb{K}^m \to \mathbb{K}$ in the φ-twisted complex have rank 1, and the rank of the central map is $\geq m-1$. By dimension counting, it follows that the complex is acyclic. We can therefore calculate its torsion using Equation (3.15). For $\mathrm{C}_i^{\varphi}(A;\mathbb{Z})$ choose the basis c_i, and as b_i use the following sequences: $b_0 = \emptyset$, $b_1 = y_s$ (the s-th vector of c_1), $b_2 = c_2 \setminus \{x_r\}$ (where x_r is the r-th vector of c_2) and $b_3 = c_3$.

Then, if we denote by a_{ij} the entries of the matrix $\varphi(\widetilde{M})$,

$$\begin{aligned}
\left[(\partial_1 b_1)b_0/_{c_0}\right] &= \left[\varphi(h_s-1)c_0/_{c_0}\right] = \varphi(h_s-1);\\
\left[(\partial_2 b_2)b_1/_{c_1}\right] &= \left[\left(\sum_{\substack{i_1,\ldots,i_{r-1},\\ i_{r+1}\ldots,i_m}}\left(\prod_{k\neq r} a_{i_k k}\right) y_{i_1}\cdots y_{i_{r-1}}y_{i_{r+1}}\cdots y_{i_m}\right) y_s\Big/_{c_1}\right]\\
&= \left[(\varphi(\Delta^{r,s})\,(c_1\setminus\{y_s\})\,y_s)\Big/_{c_1}\right]\\
&= \varphi(\Delta^{r,s})(-1)^{m-s}\left[c_1/_{c_1}\right]\\
&= (-1)^{m-s}\varphi(\Delta^{r,s});\\
\left[(\partial_3 b_3)b_2/_{c_2}\right] &= \left[\left(\sum_{\widetilde{r}=1}^{m}\varphi(g_{\widetilde{r}}-1)x_{\widetilde{r}}\,(c_2\setminus\{x_{\widetilde{r}}\})\right)\Big/_{c_2}\right]\\
&= (-1)^{r-1}\varphi(g_r-1);\\
\left[b_3/_{c_3}\right] &= 1.
\end{aligned}$$

By Equation (3.15) the torsion of the φ-twisted complex is

$$\tau(\mathrm{C}_*^{\varphi}(A;\mathbb{Z})) = (-1)^{m+r+s+1}\,\frac{\varphi(\Delta^{r,s})}{\varphi(h_s-1)\varphi(g_r-1)}.$$

The Reidemeister-Franz φ-torsion (with respect to the standard homological orientation of A and to the Euler structure $\mathfrak{s}$ defined by the chosen fundamental class e) is then

$$\tau^{\varphi}(A,\mathfrak{s}) = (-1)^{m+r+s+1}\,\tau_0(A,c)\,\frac{\varphi(\Delta^{r,s})}{\varphi(h_s-1)\varphi(g_r-1)},$$

where c is the basis of $\mathrm{C}_*(A;\mathbb{Z})$ given by the cells.

Finally, recall that the matrix $\widetilde{M}$ is the matrix obtained by multiplying each column of the matrix of Fox derivatives by an element l_i. Let l be the product of all l_i except l_r, let $\widehat{M}$ denote the matrix of Fox derivatives of the chosen relations b_i (cf. Remark 3.31), and let $\widehat{\Delta}^{r,s}$ denote the determinant of the minor (s, r) of $\widehat{M}$). Then

$$\tau^{\varphi}(A, \mathfrak{s}) = (-1)^{m+r+s+1}\,\tau_0(A, c)\,\varphi(l)\,\frac{\varphi(\widehat{\Delta}^{r,s})}{\varphi(h_s - 1)\varphi(g_r - 1)},$$

so by Equation (3.21)

$$\tau^{\varphi}(A, l^{-1}\cdot \mathfrak{s}) = (-1)^{m+r+s+1}\,\tau_0(A, c)\,\frac{\varphi(\widehat{\Delta}^{r,s})}{\varphi(h_s - 1)\varphi(g_r - 1)}. \tag{3.26}$$

The calculation made in this subsection may be summarized in the following lemma.

Lemma 3.32. *Let Y be a 3-dimensional closed connected oriented manifold, and let A be a cellular decomposition of Y in one 0-cell, m 1-cells $e_1, \dots e_m$, m 2-cells $f_1, \dots, f_m$, and one 3-cell. Suppose that every cell is endowed with an orientation. Choose relations b_i that represent in homotopy the boundaries of the 2-cells (cf. Remark 3.31). Then*

$$\pi_1(A) = \langle e_1, \dots, e_m \,|\, b_1, \dots, b_m\rangle$$

is a presentation of the fundamental group of A.

Let $\widehat{M} = (\partial_i b_j)_{i,j}$ be the matrix of Fox derivatives of the relations, and let $\widehat{\Delta}^{r,s}$ denote the determinant of the minor of $\widehat{M}$ obtained by deleting the s-th row and the r-th column.

Let h_i be the homology class in $H = \mathrm{H}_1(A;\mathbb{Z})$ represented by the 1-cell e_i, and let $g_i \in H$ be the homology class of a loop in A that intersects once positively the 2-cell f_i and is disjoint from the other 2-cells.

Then, there exists an Euler structure $\mathfrak{t}$ on Y (which depends on the choice of the relations $b_1, \dots, b_m$) such that, for each ring homomorphism $\varphi\colon\ \mathbb{Z}[H] \to \mathbb{K}$, with $\varphi(g_r - 1) \neq 0$, $\varphi(h_s - 1) \neq 0$ and $\varphi(\widehat{\Delta}^{r,s}) \neq 0$, the φ-torsion is given by

$$\tau^{\varphi}(A, \mathfrak{t}) = (-1)^{m+r+s+1}\,\tau_0(A, c)\,\frac{\varphi(\widehat{\Delta}^{r,s})}{\varphi(h_s - 1)\,\varphi(g_r - 1)}, \tag{3.27}$$

where the homological orientation of the torsion is always the standard orientation of $\mathrm{H}_*(A;\mathbb{R})$ (which exists since A is odd-dimensional) and c is the basis of $\mathrm{C}_*(A;\mathbb{R})$ given by the cells.

Remark. The Euler structure $\mathfrak{t}$ in the statement of Lemma 3.32 is the Euler structure $l^{-1}\cdot\mathfrak{s}$ that appear in Equation (3.26). The torsion relative to the other Euler structure can be obtained by Equation (3.21).

Chapter 4

Infinite families of non-quasi-alternating thin knots

In this section families of non-quasi-alternating thin knots will be detected among Kanenobu's knots.

First, some special families of knots (among Kanenobu's knots) will be defined and it will be proved that knots belonging to the same family have the same homological invariants (Khovanov, odd-Khovanov and Knot Floer homology). Hence, if one knot in the family is thin, so are the others.

Then, the calculation of the Turaev torsion will show that only finitely many knots in each family satisfy Equation (2.8), so infinitely many must be non-quasi-alternating. The calculation of the Turaev torsion is made starting from a Heegaard diagram for the branched double cover of each knot in the family, which gives a cellular structure to the branched double cover and a presentation of its fundamental group.

Finally, the discovery of thin knots in some of these families will prove that the family that they belong to contain infinite non-quasi-alternating thin knots.

This technique to find non-quasi-alternating thin knots is taken from [GW11], where Greene and Watson exhibit a family of such knots.

4.1 Kanenobu's knots

Definition 4.1. Kanenobu's knot $K_{p,q}$, with $p, q \in \mathbb{Z}$, is the knot illustrated in Figure 4.1 if $p > 0$ and $q > 0$. If $p < 0$ (resp. $q < 0$), instead of p (resp. q) right twists there are $|p|$ (resp. $|q|$) left twists.

Kanenobu's knots satisfy the following properties.

Lemma 4.2. *For each $p, q \in \mathbb{Z}$, $K_{-p,-q}$ is equivalent to $K^r_{p,q}$, the obverse of $K_{p,q}$ (cf. Definition 1.9).*

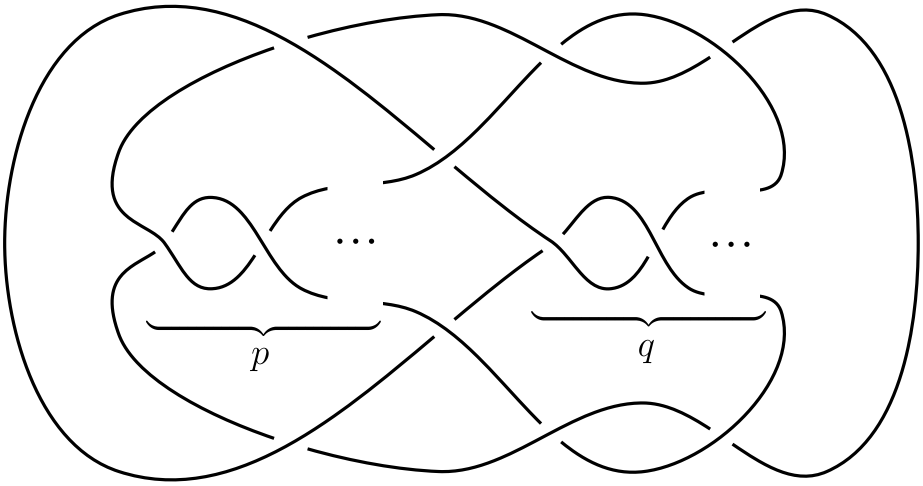


Figure 4.1: Kanenobu's knot $K_{p,q}$, where p and q are the numbers of half-twists (in the picture p and q are assumed to be positive).

Proof. A reflection along the central horizontal axis carries the diagram for $K_{p,q}$ in Figure 4.1 to the diagram for $K_{-p,-q}$. $\square$

Theorem 4.3 ([GW11, Theorems 7, 9, 10])**.** *For all $p, q \in \mathbb{Z}$,*

$$\mathrm{Kh}(K_{p,q}) \cong \mathrm{Kh}(K_{p+1,q-1}),$$
$$\mathrm{Kh}^{\mathrm{odd}}(K_{p,q}) \cong \mathrm{Kh}^{\mathrm{odd}}(K_{p+1,q-1}),$$
$$\widehat{\mathrm{HFK}}(K_{p,q}) \cong \widehat{\mathrm{HFK}}(K_{p+2,q}) \cong \widehat{\mathrm{HFK}}(K_{p,q+2}).$$

Corollary 4.4. *For each $p_0, q_0 \in \mathbb{Z}$, consider the family of Kanenobu's knots $\{K_{p_0+2n,q_0-2n}\}_{n\in\mathbb{Z}}$. If one of the knots in the family is thin (cf. Definition 1.70), so are all the knots in the family.*

Proof. The statement follows at once from Theorem 4.3. $\square$

Thin knots satisfy the following property, which is a well-known consequence of [OS05, Theorem 1.1] (see for instance [GW11, Proposition 11]).

Theorem 4.5. *The branched double cover of a thin knot is an L-space.*

Now, suppose that $\{K_{p_0+2n,q_0-2n}\}_{n\in\mathbb{Z}}$ is a family of thin knots with bounded determinants and that for each n there exists a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{t}_n$ such that the correction terms $d(\Sigma(K_{p_0+2n,q_0-2n}), \mathfrak{t}_n)$ tend to $-\infty$ as $n \to \infty$. Then, by Theorem 2.10 the knots K_{p_0+2n,q_0-2n} are non-quasi-alternating for $n \gg 0$.

Corollary 4.8 below shows that for such knots the above correction terms tend to $-\infty$ if and only if the Turaev torsion does.

Theorem 4.6 ([Rus04, Theorem 3.4]). *If Y is an L-space and $\mathfrak{t}$ is a Spin$^{\mathbb{C}}$ structure on Y, then*

$$d(Y,\mathfrak{t}) = 2\,\tau\left(Y,\mathfrak{t},1_{H_1(Y;\mathbb{Z})}\right) - \lambda(Y).$$

$\lambda(Y)$ denotes the Casson-Walker invariant of the manifold Y. If Y is the branched double cover of a link, it can be explicitly calculated, as stated in the following theorem.

Theorem 4.7 ([Mul93, Theorem 5.1]). *Let L be a link such that* $\det L \neq 0$. *Then*

$$\lambda(\Sigma(L)) = -\frac{V_L'(-1)}{6\,V_L(-1)} + \frac{\sigma(L)}{4}.$$

Corollary 4.8. *Consider the family of Kanenobu's knots $\{K_{p_0+2n,q_0-2n}\}$, and suppose that one of them is thin. Then there exists a constant $\lambda \in \mathbb{Q}$ such that for each $n \in \mathbb{Z}$ and for each* $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(K_{p_0+2n,q_0-2n}))$

$$d(\Sigma(K_{p_0+2n,q_0-2n}),\mathfrak{t}) = 2\,\tau(\Sigma(K_{p_0+2n,q_0-2n}),\mathfrak{t},1) - \lambda.$$

Proof. By Corollary 4.4 all the knots K_{p_0+2n,q_0-2n} are thin, so the branched double covers $\Sigma(K_{p_0+2n,q_0-2n})$ are L-spaces (cf. Theorem 4.5). Thus, Theorem 4.6 implies that

$$d(\Sigma(K_{p_0+2n,q_0-2n}),\mathfrak{t}) = 2\,\tau\left(\Sigma(K_{p_0+2n,q_0-2n}),\mathfrak{t},1\right) - \lambda(\Sigma(K_{p_0+2n,q_0-2n})).$$

The thing that is still to be proved is that $\lambda(\Sigma(K_{p_0+2n,q_0-2n}))$ is in fact a constant λ. The determinant of a knot $K_{p,q}$ is always 25 (an explicit computation will be done in Section 4.2.4, and the result is stated in Corollary 4.14), so Theorem 4.7 can be applied, and the Casson-Walker invariant is

$$\lambda(\Sigma(K_{p_0+2n,q_0-2n})) = -\frac{V'_{K_{p_0+2n,q_0-2n}}(-1)}{6\,V_{K_{p_0+2n,q_0-2n}}(-1)} + \frac{\sigma(K_{p_0+2n,q_0-2n})}{4}.$$

By Theorems 4.3 and 1.69 the Jones polynomials are the same. Moreover, $K_{p,q}$ is always ribbon (it is symmetric along the horizontal axis in Figure 4.1), hence algebraically slice, so the signature $\sigma(K_{p,q})$ vanishes. Thus, $\lambda(\Sigma(K_{p_0+2n,q_0-2n}))$ is in fact a constant λ, and the corollary is proved. □

A consequence of Corollary 4.8 is that, if $\{K_{p_0+2n,q_0-2n}\}_{n\in\mathbb{Z}}$ is a family of thin knots with bounded determinants, then, if for each n there exists a Spin$^{\mathbb{C}}$ structure $\mathfrak{t}_n$ such that the torsion $\tau(\Sigma(K_{p_0+2n,q_0-2n}),\mathfrak{t}_n,1)$ tends to $-\infty$ as $n \to \infty$ (resp. $n \to -\infty$), then the knots K_{p_0+2n,q_0-2n} for $n \gg 0$ (resp. $n \ll 0$) are not quasi-alternating (cf. Theorem 2.10).

4.2 The branched double cover $\Sigma(K_{p,q})$

In this section a Heegaard diagram of the manifold $\Sigma(K_{p,q})$ is given. This presentation of $\Sigma(K_{p,q})$ will naturally endow $\Sigma(K_{p,q})$ with a cellular structure and will lead to a presentation of $\pi_1(\Sigma(K_{p,q}))$ and to a calculation of the Turaev torsion.

4.2.1 Heegaard diagrams

Definition 4.9. Let Y be a closed connected oriented 3-manifold. A **Heegaard splitting** of Y is a decomposition of Y into two connected handlebodies U_1 and U_2, whose boundaries are identified via a diffeomorphism.

Definition 4.10. A **Heegaard pair** is a pair (S, α), where S is a closed connected oriented surface of genus g and α is a collection of closed simple curves in S which satisfies the following three conditions:

1. the number of curves in α is g;
2. the curves in α are disjoint;
3. after cutting S along the curves, the surface is still connected (i.e. it is a 2-dimensional sphere S^2 with $2g$ open disks removed).

A Heegaard pair (S, α) naturally defines a manifold U_α, which is obtained by taking S, attaching g 2-handles to S in such a way that the attaching spheres are the curves in α, and finally eliminating the boundary through the attachment of a 3-handle (note that a single 3-handle is required because condition 3. implies that the boundary of the manifold obtained after attaching all the 2-handles is a sphere S^2). U_α can be given a natural structure of 1-handlebody with only one 0-handle just by reversing all the handles (i.e. by taking the dual decomposition).

Definition 4.11. A **Heegaard diagram** is a triple (S, α, β) such that (S, α) and (S, β) are Heegaard pairs.

Any Heegaard diagram (S, α, β) defines a 3 manifold together with a Heegaard splitting of it. The two pairs (S, α) and (S, β) define handlebodies U_α and U_β with common boundary S. Thus, the manifold $\overline{U_\alpha} \cup_S U_\beta$ has a natural Heegaard splitting as the union of $\overline{U_\alpha}$ and U_β. Such splitting will be called the Heegaard splitting associated with the diagram (S, α, β) ($\overline{U_\alpha}$ denotes the handlebody U_α with opposite orientation). For example, the Heegaard diagram in Figure 4.2 is a Heegaard diagram for S^3.

4.2.2 A Heegaard diagram for $\Sigma(K)$

In this subsection a Heegaard diagram for the branched double cover of a knot K in S^3 is described, following [Gre08, Sect. 3.2].

Consider a diagram D of K, which gives a tetravalent graph on the plane. Colour the regions defined by the graph in white or black in a chessboard fashion, so that the unbounded region is white (see Figure 4.3). The union of the black regions gives rise to a (generally non-orientable) surface C, which is embedded in S^3 in such a way that its boundary is K: such a surface can be constructed by taking the disjoint union of the black regions and attaching at each crossing of the diagram D a half-twisted band to the

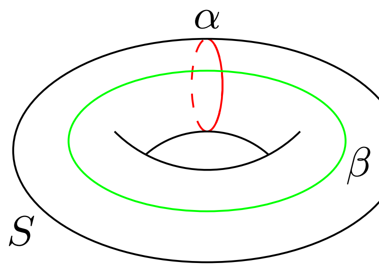


Figure 4.2: A Heegaard diagram (S, α, β) for S^3. α is the set containing only the red curve, whereas β is the set containing only the green curve.

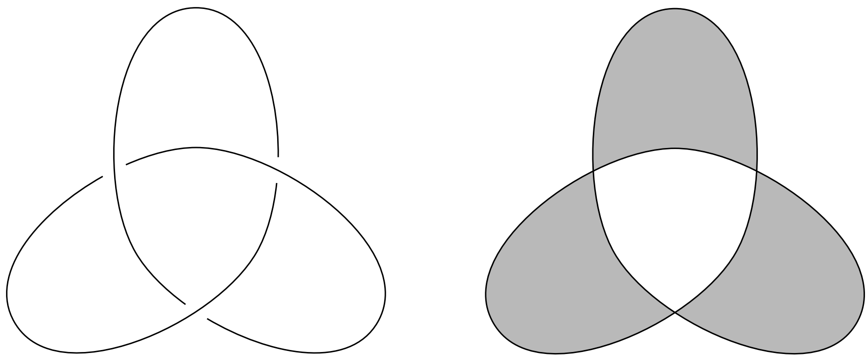


Figure 4.3: The diagram of a trefoil knot and its chessboard colouring.

adjacent black regions (the direction of the half-twist depends on the type of the crossing).

Now consider the product $[-1, 1] \times C$, and for each $p \in K$ collapse the segment $[-1, 1] \times \{p\}$ to the point $(0, p)$: the result is a connected 1-handlebody U, naturally embedded in S^3, and such that $\{0\} \times C \subseteq U$ is identified with C itself. U is a connected 1-handlebody because it is a regular neighbourhood of the black graph $\mathcal{B}$ (i.e. the planar graph obtained by drawing a node for each black region and an arc connecting two black regions for each crossing in which the two black regions meet). Let S denote the boundary of U. Figure 4.4 shows the surface S in case K is the trefoil knot in Figure 4.3.

$S \setminus K$ is a double cover of the open surface $\mathring{C}$. It is connected if C is not orientable, otherwise it is disconnected. In any case, there exists a map $\iota : S \to S$ that on $S \setminus K$ acts as the non-trivial deck transformation of the cover and on K is the identity. Such ι is the quotient of the map which sends $(\varepsilon, x) \in \{\pm 1\} \times C$ to $(-\varepsilon, x)$. It is clear that $\iota \circ \iota = \mathrm{id}_S$. The branched

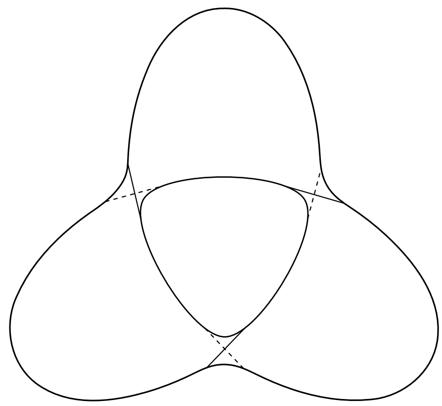


Figure 4.4: The picture shows the surface S, which is the boundary of the 1-handlebody U constructed by 'thickening' the black surface C.

double cover of K can be constructed by taking two disjoint copies of $S^3 \setminus U$ and gluing them along the boundary through the map ι. Let us prove it. Let $p : \pi_1(S^3 \setminus K) \to \mathbb{Z}_2$ be the map that associates to each $[\gamma] \in S^3 \setminus K$ the (mod 2) intersection number with the surface C (note that this map is defined even if C is non-orientable). The kernel of p consists of the curves that intersect C an even number of times. The manifold constructed above by gluing the two copies of $S^3 \setminus U$ admits a projection on S^3 using the fact that $S^3 \setminus U \cong S^3 \setminus C$. such projection is a branched cover whose branch set is K and whose associated cover is the one corresponding to the subgroup $\ker p$ of $\pi_1(S^3 \setminus K)$ (because a loop in S^3 lifts to a closed path in the cover if and only if it pierces C an even number of times). Hence, by Lemma 1.56, the resulting manifold is $\Sigma(K)$.

Let $\alpha_1, \dots, \alpha_m$ be the simple closed curves obtained by intersecting S with the white regions except the unbounded one, and for each i let β_i be a simple closed curve on S intersecting α_i once and disjoint from the other α_j's (cf. Figure 4.5). If α denotes the set of the curves α_i and β denotes the set of the curves β_i (which may be supposed disjoint), then $(\overline{S}, \beta)$ is a Heegaard pair defining the handlebody $U_\beta = U$ and (S, α) is a Heegaard diagram defining the complementary handlebody $U_\alpha = S^3 \setminus U_\beta$. For each i, let γ_i be the image of α_i under ι. Then, if γ is the set of the curves γ_i, (S, γ) is still a Heegaard pair for the 1-handlebody U_α. Since $\Sigma(K)$ is the manifold obtained by quotienting two copies of U_α on the boundary under the map $\iota : S \to S$, (S, α, γ) is a Heegaard diagram for $\Sigma(K)$.

Let us describe in detail the curve γ_i: it coincides with α_i away from the crossings of K, whereas at each crossing it undergoes a twist in the same direction as the crossing. However, for the sake of clarity, we can suppose that away from the crossings γ_i is just parallel to α_i, and not coincident with it (cf. Figures 4.5 and 4.6).

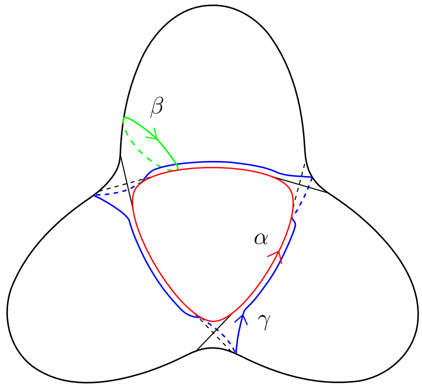


Figure 4.5: The picture shows the α, β and γ curves for the diagram of a trefoil knot as in Figure 4.3. In each one of the three cases there is only one curve belonging to the set α, β or γ. The red curve is the one belonging to the α set, the green curve is the one belonging to the β set, and the blue curve is the one belonging to the γ set.

Remark. A standard orientation of the α, β and γ curves can be chosen as follows (see also Figures 4.5 and 4.6). The α curves are the intersection of bounded white regions with the surface S; since they lie in the plane of the diagram D, they can be endowed with the counterclockwise orientation. The surface S comes with an orientation since it is the boundary of the 1-handlebody U_β. Hence, the intersection of two oriented curves on S has a sign. The condition that guarantees a choice of the orientation of β_i is then

$$\#(\alpha_i \cap \beta_i) = +1. \tag{4.1}$$

Finally, the orientation of γ_i is induced by the orientation of α_i through the map ι.

4.2.3 A presentation of $\pi_1(\Sigma(K))$

In the last subsection a Heegaard diagram (S, α, γ) for the double branched cover of a knot K was given starting from a diagram of the knot. Such a Heegaard diagram gives a Heegaard splitting of the manifold $\Sigma(K)$ into two 1-handlebodies $\overline{U_\alpha}$ and U_γ. This splitting also induces a handle decomposition of $\Sigma(K)$: indeed $\Sigma(K)$ is obtained from the 1-handlebody $\overline{U_\alpha}$ by attaching 2-handles along the γ curves and finally a 3-handle.

Since a handle is a thickened version of a cell, the handle decomposition induces a cellular decomposition of $\Sigma(K)$, where (roughly speaking) the λ-cells are given by the cores of the λ-handles. The curves $\beta_1, \dots, \beta_m$ are homotopic in $\Sigma(K)$ respectively to the 1-cells $e_1, \dots, e_m$ of this decomposition, whereas the γ curves are homotopic to the boundaries of the 2-cells

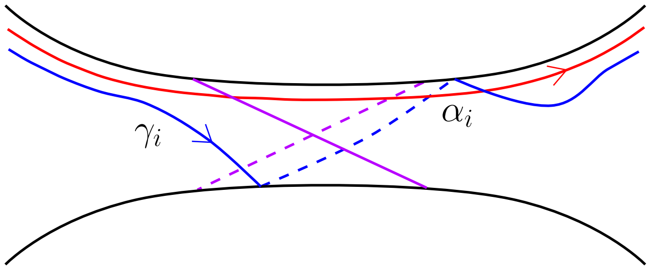


Figure 4.6: The picture shows the curves α_i (in red) and γ_i (in blue) near a crossing. For the sake of clarity they have been slightly shifted, in such a way that they are transverse. The purple lines are the segments of K that constitute the crossing, so they are fixed under the action of ι.

$f_1, \dots, f_m$. Moreover, since the β and the γ curves are oriented (as explained at the end of Section 4.2.2), an orientation is induced on the cells $e_1, \dots, e_m, f_1, \dots, f_m$.

Thus, a presentation of $\pi_1(\Sigma(K))$ is obtained as follows: the oriented 1-cells $e_1, \dots, e_m$ (or, analogously, the curves $\beta_1, \dots, \beta_m$) are taken as generators, and, in order to obtain a relation b_j, we should consider in $\overline{U_\alpha}$ the curve γ_j (lying in the same homotopy class as ∂f_j) and express its homotopy class in terms of the 1-cells e_i (or, equivalently, in terms of the β curves). Since the β curves represent the 1-handles, we shall record which 1-handles γ_j overpasses. It passes over the 1-handle corresponding to β_i if it meets transversally once the belt sphere of the 1-handle (cf. Figure 4.7). But the belt sphere of the 1-handle is exactly the curve α_i, because it is defined by the condition that α_i intersects once β_i and never intersects the other β curves. Hence, to obtain the relation b_j it is sufficient to record a e_i or a e_i^{-1} when γ_j intersects α_i. The choice between e_i and e_i^{-1} depends on the sign of the intersection: recall that the orientation of β_i is defined in such a way that in S

$$\#(\alpha_i \cap \beta_i) = +1.$$

Hence, γ_j goes over the i-th 1-handle in the same direction as β_i if and only if on S the sign of the intersection between α_i and γ_j is positive.

The intersections between γ_j and the α curves are concentrated around the crossings of K. Focus on a particular crossing c, and give it a sign $\varepsilon(c)$ according to Figure 4.8. Suppose that the adjacent white regions correspond to the α curves α_i and α_j. If c is a negative crossing (i.e. if $\varepsilon(c) = -1$), then γ_j first intersects α_j and then α_i (see the picture on the left in Figure 4.9), and the incidence relations in S are:

$$\#(\alpha_j \cap \gamma_j) = +1; \tag{4.2a}$$
$$\#(\alpha_i \cap \gamma_j) = -1. \tag{4.2b}$$

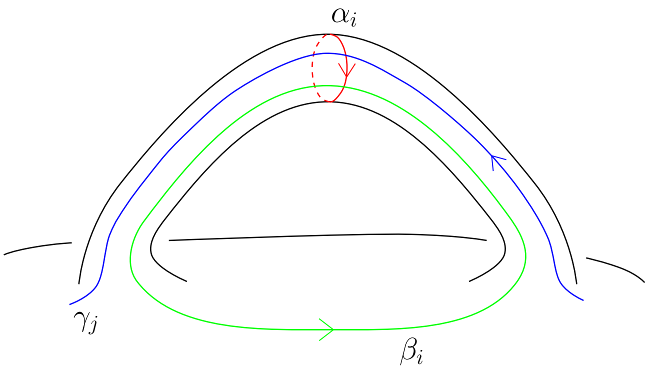


Figure 4.7: The picture shows the 1-handle corresponding to β_i in $\overline{U_\alpha}$. The belt sphere of the 1-handle is the curve α_i. The fact that γ_j goes over the handle can be detected from the fact that γ_j intersects α_i.

If instead c is a positive crossing (i.e. if $\varepsilon(c) = +1$), then γ_j first intersects α_i and then α_j (see the picture on the right in Figure 4.9), and the incidence relations in S are:

$$\#(\alpha_i \cap \gamma_j) = +1; \tag{4.3a}$$
$$\#(\alpha_j \cap \gamma_j) = -1. \tag{4.3b}$$

Hence, when we write the relation b_j, we have to add $(e_i e_j^{-1})^{\varepsilon(c)}$ if γ_j passes through the crossing c.

Suppose now that the adjacent white regions are the one defined by α_j and the unbounded one. In this case α_i does not exists, and the relation that must be recorded is simply $e_j^{-\varepsilon(c)}$.

The relation b_j can now be written simply following the curve γ_j and writing the relations recorded at each crossing sequentially. For example, in the case of the trefoil knot as in Figure 4.3, the unique relation b is given by

$$b = e^3,$$

as Figure 4.5 illustrates.

To summarize the discussion above, the presentation of $\pi_1(\Sigma(K))$ can be recovered directly from the diagram of K as follows. First, we have to construct a graph associated to the diagram, which is called reduced white graph.

Definition 4.12. Let D be a diagram of a knot. Colour the plane in a chessboard fashion so that the unbounded region is white. Call the white regions $w_1, \ldots, w_m, w_{\text{unb}}$ (w_{unb} is the unbounded region). Now consider the white graph $\mathcal{W}$ (i.e. the graph whose nodes are the white regions and whose arcs connecting w_i and w_j are the crossings where w_i and w_j are adjacent), and remove the node corresponding to the unbounded region from it (but

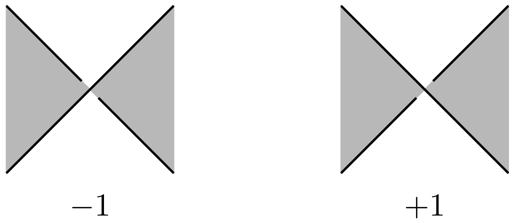


Figure 4.8: If D is a diagram of a knot with a chessboard colouring, assign to each crossing c a sign $\varepsilon(c)$ as in the picture.

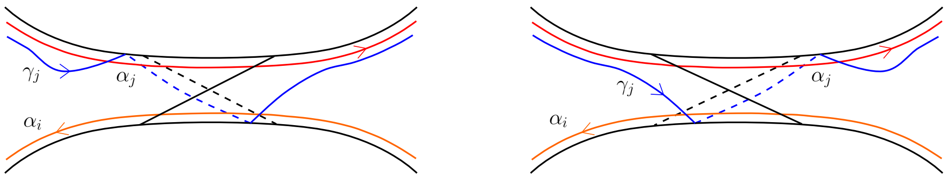


Figure 4.9: The picture on the left shows that at a negative crossing the curve γ_j first intersects α_j and then intersects α_i, and the signs of the intersections are as in Equations (4.2), whereas the picture on the right shows the case of a positive crossing, where the curve γ_j first intersects α_i and then intersects α_j, and the signs of the intersections in this case are as in Equations (4.3).

do not remove the arcs). Then label each arc with a sign $+$ or $-$ depending on the sign of the corresponding crossing (cf. Figure 4.8), and denote the resulting graph by $\widetilde{\mathcal{W}}$.

$\widetilde{\mathcal{W}}$ is called the **reduced white graph** of the diagram D.

Figure 4.10 shows the reduced white graph for the diagram of the trefoil knot in Figure 4.3.

For each node w_j of the reduced white diagram, consider the arcs of the diagram that exit from w_j counterclockwise. For each arc c connecting w_j to w_i record a 'word' $(e_i e_j^{-1})^{\varepsilon(c)}$ (where $\varepsilon(c)$ denotes the sign assigned to the arc). If the arc was connecting w_j to the removed node (the one corresponding to the unbounded region), then the word to be recorded reduces to $e_j^{-\varepsilon(c)}$. The relation b_j is obtained by writing the recorded words in the order the arcs exiting from w_j appear if you go counterclockwise around the node (the point you start going counterclockwise simply changes b_j by a conjugation). In the case of the trefoil knot, the only relation given by the reduced white graph in Figure 4.10 is e_1^3.

A presentation of $\pi_1(\Sigma(K))$ is then given by

$$\pi_1(\Sigma(K)) = \langle e_1, \ldots, e_m \,|\, b_1, \ldots, b_m \rangle.$$

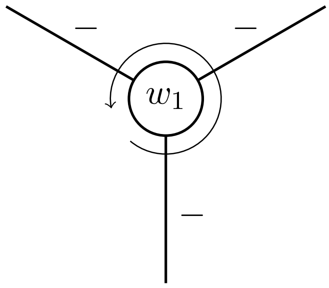


Figure 4.10: The reduced white graph $\widetilde{\mathcal{W}}$ for the diagram of trefoil knot in Figure 4.3.

4.2.4 The case of $\Sigma(K_{p,q})$

Suppose now that K is the Kanenobu's knot $K_{p,q}$. Consider the diagram D of K as in Figure 4.1. Then, the reduced white graph is as in Figure 4.11.

The relations given by the reduced white graph are the following:

$$b_1 = \left(e_2e_1^{-1}\right)^p e_4e_1^{-2}; \tag{4.4a}$$
$$b_2 = e_2e_3^{-1}\left(e_1e_2^{-1}\right)^p e_2; \tag{4.4b}$$
$$b_3 = \left(e_4e_3^{-1}\right)^q e_3e_2^{-1}e_3^2; \tag{4.4c}$$
$$b_4 = e_1e_4^{-1}\left(e_3e_4^{-1}\right)^q e_4^{-2}. \tag{4.4d}$$

A presentation of $\pi_1(\Sigma(K_{p,q}))$ is then given by

$$\pi_1(\Sigma(K_{p,q})) = \langle e_1, e_2, e_3, e_4 \,|\, b_1, b_2, b_3, b_4\rangle.$$

By abelianizing the relations in Equations (4.4) we obtain the following presentation matrix for $H = \mathrm{H}_1(\Sigma(K_{p,q}); \mathbb{Z})$:

$$M_{p,q} = \begin{pmatrix} -p-2 & p & 0 & 1 \\ p & -p+2 & -1 & 0 \\ 0 & -1 & -q+3 & q \\ 1 & 0 & q & -q-3 \end{pmatrix}. \tag{4.5}$$

Let us calculate a Hermite decomposition of $M_{p,q}$, which will give the structure of the $\mathbb{Z}$-module H. First reverse the order of the columns (which means reversing the order of the relations):

$$M_{p,q} \sim \begin{pmatrix} 1 & 0 & p & -p-2 \\ 0 & -1 & -p+2 & p \\ q & -q+3 & -1 & 0 \\ -q-3 & q & 0 & 1 \end{pmatrix}.$$

Then perform the following column operations (that do not change the generators of the presentation): add to the third column the first one multiplied by $-p$ and the second one multiplied by $(-p+2)$, and add to the fourth

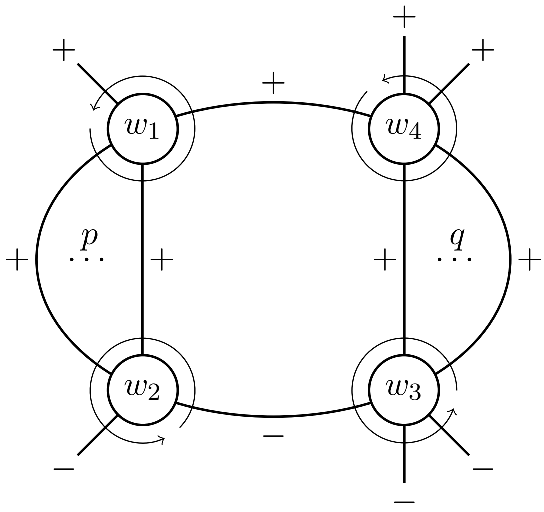


Figure 4.11: The reduced white graph $\widetilde{\mathcal{W}}$ for the diagram of Kanenobu's knot $K_{p,q}$ in Figure 4.1.

column the first one multiplied by $(p+2)$ and the second one multiplied by p. Then

$$M_{p,q} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ q & -q+3 & -3p-2q+5 & 3p+2q \\ -q-3 & q & 3p+2q & -3p-2q-5 \end{pmatrix}.$$

Now add the third column to the fourth one:

$$M_{p,q} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ q & -q+3 & -3p-2q+5 & 5 \\ -q-3 & q & 3p+2q & -5 \end{pmatrix}. \tag{4.6}$$

The presentation matrix in the form of Equation (4.6) already gives some information, which is summarized in the following proposition.

Proposition 4.13. *Let $M_{p,q}$ be the matrix defined by Equation (4.5). Then*

1. *the Hermite decomposition of the matrix $M_{p,q}$ depends only on $[p]$ and $[q]$ in $\mathbb{Z}_5$;*

2. *for each $p, q \in \mathbb{Z}$, $\det M_{p,q} = 25$.*

Proof. Consider the presentation matrix in the form of Equation (4.6). If p or q are changed by a multiple of 5, then by adding or subtracting the fourth column to the others we can go back to the matrix we had before changing p and q. Hence the first point is proved.

The second point follows from an easy computation. □

Incidentally, the second point of Proposition 4.13, combined with Lemma 1.58, proves the following corollary.

Corollary 4.14. *For each $p, q \in Z$, $\det K_{p,q} = 25$.*

Now assume that $[p] \neq [q] \in \mathbb{Z}_5$. Then the numbers $-3p-2q+5$ and 5 are coprime, so there exist integers s and t such that

$$(-3p-2q+5)s + 5t = 1. \tag{4.7}$$

Change the matrix in Equation (4.6) by right multiplication by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & -5 \\ 0 & 0 & t & -3p-2q+5 \end{pmatrix},$$

which is a matrix invertible over $\mathbb{Z}$ (since its determinant is 1). The result of the product is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ q & -q+3 & 1 & 0 \\ -q-3 & q & -1+5s & -25 \end{pmatrix}.$$

Now multiply the second and the last column by -1. Then add to the first column the third one multiplied by $-q$ and add to the second column the third one multiplied by $(-q+3)$. The final result is

$$M_{p,q} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3-5sq & -3-5(q-3)s & -1+5s & 25 \end{pmatrix}, \tag{4.8}$$

which is the Hermite form of $M_{p,q}$ if $[p] \neq [q]$ in $\mathbb{Z}_5$. Then in this case the group $H = \mathrm{H}_1(\Sigma(K_{p,q}); \mathbb{Z})$ is isomorphic to $\mathbb{Z}_{25}$. We shall use multiplicative notation to denote the operation on $\mathbb{Z}_{25}$. Call t the generator of H induced by e_4. Then, the columns of the matrix in Equation (4.8) turn into the relations

$$[e_1] = t^{3+5A_1} \tag{4.9a}$$
$$[e_2] = t^{3+5A_2} \tag{4.9b}$$
$$[e_3] = t^{1+5A_3} \tag{4.9c}$$
$$[e_4]^{25} = t^{25} = 1, \tag{4.9d}$$

where

$$\begin{aligned} A_1 &\equiv qs && \pmod 5\\ A_2 &\equiv (q-3)s && \pmod 5\\ A_3 &\equiv -s && \pmod 5. \end{aligned}$$

Note that, since $t^{25} = 1$, the numbers A_1, A_2 and A_3 are important only for their modulo 5 class. Moreover, since s and 5 are coprime (otherwise Equation (4.7) would not be true), the following equations hold:

$$A_1 \not\equiv A_2 \pmod 5 \tag{4.10a}$$

$$A_3 \not\equiv 0 \pmod 5. \tag{4.10b}$$

4.3 The Turaev torsion of $\Sigma(K_{p,q})$

In this section a computation of the Turaev torsion of $\Sigma(K_{p,q})$ is made. The Turaev torsion will distinguish Kanenobu's knots $K_{p,q}$ and $K_{p',q'}$ (unless (p,q) is either (p',q') or (q',p')) and it will prove (under certain conditions) the non-$\mathcal{Q}A$-ness of $K_{p,q}$.

First, we will consider only Kanenobu's knots $K_{p,q}$ such that

$$p \not\equiv q \pmod 5.$$

In terms of the diagram in Figure 4.12 this means that we are not considering the knots lying on the green lines.

If $p \not\equiv q \pmod 5$, then $H = \mathrm{H}_1(\Sigma(K_{p,q});\mathbb{Z})$ is isomorphic to Z_{25} (since a presentation matrix for $\mathrm{H}_1(\Sigma(K_{p,q});\mathbb{Z})$ is the one in Equation (4.8)). Recall that $t = [e_4]$ is a generator of H. Let φ_0, φ_1 and φ_2 be the maps defined by

$$\begin{aligned} \varphi_j :\quad \mathbb{Q}[H] &\longrightarrow \mathbb{Q}(\zeta_{5^j})\\ t &\longmapsto \zeta_{5^j} \end{aligned}$$

where $\zeta_{5^j} = e^{2\pi i/5^j} \in \mathbb{C}$. Note that $\varphi_0 : \mathbb{Q}[H] \to \mathbb{Q}$ is the augmentation map. The maps φ_0, φ_1 and φ_2 are induced by different characters $H \to \mathbb{C}^*$ (specifically, by the three different characters of H up to automorphisms of the image). By Proposition 3.28, the map

$$\varphi = (\varphi_0, \varphi_1, \varphi_2) : \mathbb{Q}[H] \to \mathbb{Q} \oplus \mathbb{Q}(\zeta_5) \oplus \mathbb{Q}(\zeta_{25})$$

is an isomorphism.

The maximal abelian torsion τ of $\Sigma(K_{p,q})$ lies in $\mathbb{Q}[H]$, and it is completely determined by the φ_j-torsions, which are the images of τ under the maps φ_j. Hence, we will calculate the φ_j-torsions.

Before starting the calculation, it is worth noting that, in view of the results of Section 4.2, for each p, q in $\mathbb{Z}$ a Heegaard diagram (S, α, γ) for

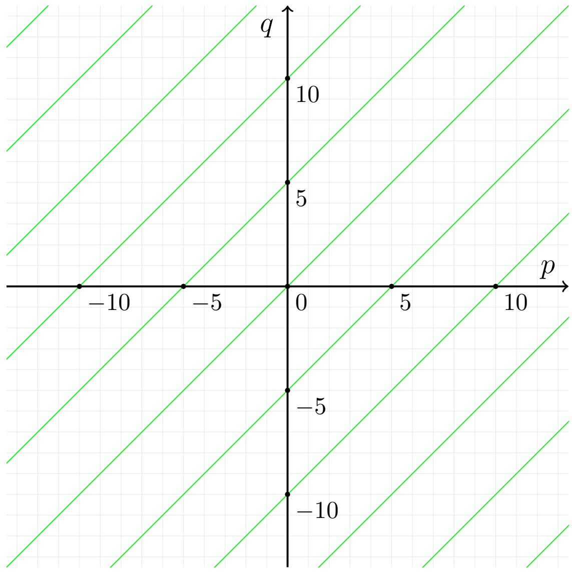


Figure 4.12: The point (p, q) of the above grid corresponds to Kanenobu's knot $K_{p,q}$. The green lines are defined by the equation $p \equiv q \pmod 5$.

$\Sigma(K_{p,q})$ is fixed. Such a Heegaard diagram gives a genus-4 Heegaard splitting of $\Sigma(K_{p,q})$, and it provides a fixed cellular decomposition of $\Sigma(K_{p,q})$. Moreover, the 1-cells $e_1, \ldots, e_4$ and the 2-cells $f_1, \ldots, f_4$ are endowed with a fixed orientation. We can also fix the relations that give a presentation of $\pi(\Sigma(K_{p,q}))$ as the ones in Equations (4.4).

In order to compute the φ_j-torsion (for $j = 1, 2$), we will apply Lemma 3.32. We will choose $r = s = 4$. The two homology classes h_4 and g_4 that appear in the statement of Lemma 3.32 are respectively $[e_4] = t$ and $\left[e_4^{-1}\right] = t^{-1}$. Indeed, g_4 is the homology class of a curve piercing (with one positive intersection) the cell f_4 and disjoint from the other 2-cells; Equation 4.1 implies that β_4 pierces the cell f_4 with one negative intersection and is disjoint from the other 2-cells, so $[g_4] = [\beta_4^{-1}] = [e_4^{-1}]$. Thus, for $j = 1, 2$ the hypotheses $\varphi_j(g_4 - 1) \neq 0$ and $\varphi_j(h_4 - 1) \neq 0$ are satisfied. The $(4, 4)$ minor of the matrix $\widehat{M}$ is

$$\widehat{\Delta}^{4,4} = \det \begin{pmatrix} [\partial_1 b_1] & [\partial_1 b_2] & [\partial_1 b_3] \\ [\partial_2 b_1] & [\partial_2 b_2] & [\partial_2 b_3] \\ [\partial_3 b_1] & [\partial_3 b_2] & [\partial_3 b_3] \end{pmatrix}, \tag{4.11}$$

where

$$\partial_1 b_1 = \partial_1 \left((e_2 e_1^{-1})^p\right) - (e_2 e_1^{-1})^p e_4 (e_1^{-1} + e_1^{-2}) \tag{4.12a}$$
$$\partial_2 b_1 = \partial_2 \left((e_2 e_1^{-1})^p\right) \tag{4.12b}$$
$$\partial_3 b_1 = 0 \tag{4.12c}$$

$$\partial_1 b_2 = e_2 e_3^{-1} \partial_1 \left((e_1 e_2^{-1})^p \right) \tag{4.12d}$$
$$\partial_2 b_2 = 1 + e_2 e_3^{-1} \partial_2 \left((e_1 e_2^{-1})^p \right) + e_2 e_3^{-1} (e_1 e_2^{-1})^p \tag{4.12e}$$
$$\partial_3 b_2 = -e_2 e_3^{-1} \tag{4.12f}$$
$$\partial_1 b_3 = 0 \tag{4.12g}$$
$$\partial_2 b_3 = -(e_4 e_3^{-1})^q e_3 e_2^{-1} \tag{4.12h}$$
$$\partial_3 b_3 = \partial_3 \left((e_4 e_3^{-1})^q \right) + (e_4 e_3^{-1})^q + (e_4 e_3^{-1})^q e_3 e_2^{-1} (1 + e_3). \tag{4.12i}$$

Note that

$$\partial_i \left((e_i e_j^{-1})^n \right) = \begin{cases} 1 + e_i e_j^{-1} + \cdots + (e_i e_j^{-1})^{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -e_j e_i^{-1} - \cdots - (e_j e_i^{-1})^{|n|} & \text{if } n < 0 \end{cases} \tag{4.13}$$

and

$$\partial_j \left((e_i e_j^{-1})^n \right) = \partial_j \left((e_j e_i^{-1})^{-n} \right). \tag{4.14}$$

We will check in Sections 4.3.2 and 4.3.3 that, for $j = 1, 2$, $\varphi_j(\widehat{\Delta}^{4,4}) \neq 0$.

Finally, the sign $\tau_0(\Sigma(K_{p,q}), c)$ is always -1 (it can be checked with an easy computation, applying Equation (3.13) to the complex $\mathrm{C}_*(\Sigma(K_{p,q}), \mathbb{R})$ with the basis c and the homology basis given by the orientation of $\Sigma(K_{p,q})$).

Hence, up to checking that $\varphi_j(\widehat{\Delta}^{4,4}) \neq 0$, Lemma 3.32 implies that there exists a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{t}_{p,q}$ such that for $j = 1, 2$

$$\tau^{\varphi_j}(\Sigma(K_{p,q}), \mathfrak{t}_{p,q}) = \frac{\varphi_j(\widehat{\Delta}^{4,4})}{\varphi_j(t-1)\, \varphi_j(t^{-1}-1)}. \tag{4.15}$$

In the next sections (4.3.1, 4.3.2 and 4.3.3) the φ_j-torsions are explicitly calculated.

4.3.1 The φ_0-torsion

The φ_0-torsion is very easy to calculate. Indeed, since φ_0 is the augmentation map, $\mathrm{C}_*^{\varphi_0}(\Sigma(K_{p,q})) \cong \mathrm{C}_*(\Sigma(K_{p,q}); \mathbb{Q})$, which is not acyclic. Hence, the φ_0-torsion $\tau^{\varphi_0}(\Sigma(K_{p,q}), \mathfrak{t})$ vanishes for all $\mathfrak{t} \in \mathrm{Spin}^{\mathbb{C}}(\Sigma(K_{p,q}))$ (cf. Definition 3.26).

4.3.2 The φ_1-torsion

The φ_1-torsion will be calculated by applying Equation (4.15), so $\varphi_1(\widehat{\Delta}^{4,4})$ has to be calculated. First note that, by Equations (4.9) and the fact that $\varphi_1(t^5) = 1$,

$$\varphi_1\left(\left[e_1 e_2^{-1}\right]\right) = 1;$$
$$\varphi_1\left(\left[e_3 e_4^{-1}\right]\right) = 1.$$

Thus, by Equations (4.13) and (4.14), in all the cases that appear in the Fox derivatives in Equations (4.12),

$$\begin{aligned}\varphi_1\left(\left[\partial_i\left((e_ie_j^{-1})^n\right)\right]\right) &= n;\\ \varphi_1\left(\left[\partial_j\left((e_ie_j^{-1})^n\right)\right]\right) &= -n.\end{aligned}$$

Then, using Equations (4.9) and the last equations, it is easy to calculate that $\varphi_1(\widehat{\Delta}^{4,4})$, which is equal to the $(4,4)$ minor of the matrix $\varphi_1(\widehat{M})$ (where $\widehat{M}$ is the matrix in Equation (4.11)):

$$\varphi_1(\widehat{\Delta}^{4,4}) = p + 2q + q\zeta_5^2 + q\zeta_5^3.$$

Thus, $\varphi_1(\widehat{\Delta}^{4,4}) \neq 0$, so the formula in Equation (4.15) can be applied to find that

$$\tau^{\varphi_1}(\Sigma(K_{p,q}), \mathfrak{t}_{p,q}) = \frac{2p+3q}{5} + \frac{-p+q}{5}\,\zeta_5^2 + \frac{-p+q}{5}\,\zeta_5^3. \tag{4.16}$$

Note that the coefficients of 1, ζ_5^2 and ζ_5^3 are non-vanishing under the assumption that $p \not\equiv q \pmod 5$.

4.3.3 The φ_2-torsion

As in the case of Section 4.3.2, we have to calculate $\varphi_2(\widehat{\Delta}^{4,4})$ and check that it is non-vanishing. Specifically, we will prove the following proposition.

Proposition 4.15. *Let p, q be integers such that $p \not\equiv q \pmod 5$. Then, $\varphi_2(\widehat{\Delta}^{4,4}) \in \mathbb{Q}(\zeta_{25})$ is non-vanishing and only depends on $[p]$ and $[q]$ in $\mathbb{Z}_5$.*

Proof. First note that, in the cases that appear in Equations (4.12),

$$\varphi_2\left(\left[e_ie_j^{-1}\right]\right) = \zeta_{25}^{5k} \tag{4.17}$$

for some k such that $[k] \in \mathbb{Z}_5 \setminus \{0\}$ (the fact that $k \not\equiv 0 \pmod 5$ is a consequence of Equations (4.10)). Then, recalling Equation (4.13) and the fact that

$$1 + \zeta_{25}^5 + \zeta_{25}^{10} + \zeta_{25}^{15} + \zeta_{25}^{20} = 0,$$

we have that the image of the Fox derivative $\partial_i\left(\left(e_ie_j^{-1}\right)^n\right)$ depends only on $[n] \in \mathbb{Z}_5$:

$$\varphi_2\left(\left[\partial_i\left(\left(e_ie_j^{-1}\right)^n\right)\right]\right) = 1 + \zeta_{25}^{5k} + \cdots + \zeta_{25}^{5k([n]-1)},$$

where $[n]$ is chosen in the set $\{1, \ldots, 5\}$. Note that

$$\varphi_2\left(\left[\partial_i\left(\left(e_ie_j^{-1}\right)^n\right)\right]\right) = 0 \iff n \equiv 0 \pmod 5.$$

By Equation (4.14), also $\varphi_2\left(\partial_j\left(\left(e_i e_j^{-1}\right)^n\right)\right)$ depends only on $[n] \in \mathbb{Z}_5$ and satisfies

$$\varphi_2\left(\left[\partial_j\left(\left(e_i e_j^{-1}\right)^n\right)\right]\right) = 0 \iff n \equiv 0 \pmod 5.$$

Since also $\varphi_2\left(\left[(e_i e_j^{-1})^n\right]\right)$ depends only on $[n] \in \mathbb{Z}_5$, we have that $\varphi_2(\widehat{\Delta}^{4,4})$ depends only on $[p]$ and $[q]$. Specifically, defining

$$x(i,j,[n]) = \varphi_2\left(\left[\partial_i\left(\left(e_i e_j^{-1}\right)^n\right)\right]\right),$$

and using the fact that

$$\varphi_2\left(\left[\partial_j\left(\left(e_i e_j^{-1}\right)^n\right)\right]\right) = x(j,i,[-n]) = -\left[e_i e_j^{-1}\right] x(i,j,[n]),$$

a computation with Mathematica (cf. [Wol03]) shows that

$$\begin{aligned}
\varphi_2(\widehat{\Delta}^{4,4}) = \zeta_{25}^{-5(1+2A_1+A_3[q])} &\left(\zeta_{25}^{5(A_2-A_1)[p]}\left(1+\zeta_{25}^{3+5A_1}\right)\right.\\
&\left.+\,\zeta_{25}^{5(1+A_1+A_2)}\,x(2,1,[p])\right) - \zeta_{25}^{-10A_3}\,x(2,1,[p])\,x(1,2,[p])\\
&\left(\zeta_{25}^{5A_3(1-[q])}\left(\zeta_{25}^{2+5A_2}+\zeta_{25}^{5A_3}+\zeta_{25}^{1+10A_3}\right) - \zeta_{25}^{2+5A_2}\,x(4,3,[q])\right)\\
&-\left(\zeta_{25}^{5(A_2-A_1)[p]}\left(1+\zeta_{25}^{3+5A_1}\right)+\zeta_{25}^{5(1+A_1+A_2)}\,x(2,1,[p])\right)\\
&\left(\zeta_{25}^{5A_3}+\zeta_{25}^{2+5A_2+5(A_1-A_2)[p]}+\zeta_{25}^{2+5A_1}\,x(1,2,[p])\right)\\
&\left(\zeta_{25}^{5A_3(1-[q])}\left(\zeta_{25}^{2+5A_2}+\zeta_{25}^{5A_3}+\zeta_{25}^{1+10A_3}\right) - \zeta_{25}^{2+5A_2}\,x(4,3,[q])\right)\\
&\zeta_{25}^{-7-5(2A_1+A_2+2A_3)}.
\end{aligned}$$

We have to check that, if $p \not\equiv q \pmod 5$, then $\varphi_2(\widehat{\Delta}^{4,4}) \in \mathbb{Z}[\zeta_{25}]$ does not vanish. Consider the projection

$$\psi : \mathbb{Z}[\zeta_{25}] \cong \mathbb{Z}[t]\Big/\left(t^{20}+t^{15}+t^{10}+t^5+1\right) \to \mathbb{Z}_5[t]\Big/\left(t^4+t^3+t^2+t+1\right),$$

induced by $t \mapsto t$. If $\psi \circ \varphi_2(\widehat{\Delta}^{4,4}) \neq 0$, so is $\varphi_2(\widehat{\Delta}^{4,4})$.

Combining Equations (4.13) and (4.17), one can show that in $\mathbb{Z}_5$

$$\psi\left(x(i,j,[n])\right) = [n].$$

Then, a calculation shows that

$$\psi \circ \varphi_2(\widehat{\Delta}^{4,4}) = [p] + 2[q] + [q]\,t^2 + [q]\,t^3.$$

$\psi \circ \varphi_2(\widehat{\Delta}^{4,4})$ vanishes if and only if all its coefficients do. However, this happens if and only if $p \equiv q \equiv 0 \pmod 5$, which is absurd because we required that $p \not\equiv q \pmod 5$. Hence,

$$\varphi_2(\widehat{\Delta}^{4,4}) \neq 0. \qquad \square$$

By Equation (4.15), a straightforward corollary of Proposition 4.15 is the following.

Corollary 4.16. *Let p, q be integers such that $p \not\equiv q \pmod 5$. Then, the φ_2-torsion $\tau^{\varphi_2}(\Sigma(K_{p,q}), \mathfrak{t}_{p,q}) \in \mathbb{Q}(\zeta_{25})$ only depends on $[p]$ and $[q]$ in $\mathbb{Z}_5$.*

The explicit form of the φ_2-torsion can also be computed using Equation (4.15), but for our purposes the previous corollary is enough.

4.4 The families of knots

This section contains the main result of the present work. Consider the family of Kanenobu's knots

$$\mathcal{F} = \{K_{p_0+2n, q_0-2n}\}_{n\in\mathbb{Z}},$$

and suppose that K_{p_0,q_0} is thin (as we will see, this is a key assumption). For each $j = 0, \dots, 4$, let us also define the subfamilies

$$\mathcal{F}_{[j]} = \{K_{p_0+10n+2j, q_0-10n-2j}\}_{n\in\mathbb{Z}} \subseteq \mathcal{F},$$

that constitute a partition of $\mathcal{F}$. The subfamily $\mathcal{F}_{[p_0-q_0]}$ is the subfamily of the knots $K_{p,q}$ such that $p \equiv q \pmod 5$. Since this case is different from the others, we will usually prefer to deal with the family

$$\widetilde{\mathcal{F}} = \mathcal{F} \setminus \mathcal{F}_{[p_0-q_0]}.$$

4.4.1 Distinguishing the knots

First, we will prove that the knots in the family $\widetilde{\mathcal{F}}$ are distinct if $n \gg 0$ or $n \ll 0$. This is a straightforward consequence of the following lemma.

Lemma 4.17. *Let p, q, r, s be integers such that $p \not\equiv q$ and $r \not\equiv s \pmod 5$. If $(p, q) \neq (r, s)$ and $(p, q) \neq (s, r)$, then the φ_1-torsions of $\Sigma(K_{p,q})$ and $\Sigma(K_{r,s})$ are different, so $K_{p,q} \neq K_{r,s}$.*

Proof. The proof requires care, firstly because the map φ_1 depends on the isomorphism between $H = \mathrm{H}_1(\Sigma(K_{\cdot,\cdot}))$ and $\mathbb{Z}_{25}$, so it is defined only up to automorphisms of $\mathbb{Q}(\zeta_5)$, and secondly because the φ_1-torsion is defined only up to multiplication by elements of H.

The automorphisms of $\mathbb{Q}(\zeta_5)$ are determined by the image of ζ_5. Let ξ_a be the automorphism specified by:

$$\begin{aligned} \xi_a : \quad \mathbb{Q}(\zeta_5) &\longrightarrow \mathbb{Q}(\zeta_5) \\ \zeta_5 &\longmapsto \zeta_5^a \end{aligned}$$

Since the torsion is defined up to automorphisms of $\mathbb{Q}(\zeta_5)$ and multiplication by ζ_5^b, it is useful to define

$$P_{p,q}(a,b) = \zeta_5^b \cdot \xi_a \left(\tau(\Sigma(K_{p,q}), \mathfrak{t}_{p,q})\right).$$

Note that $P_{p,q}(1,0) = \tau(\Sigma(K_{p,q}), \mathfrak{t}_{p,q})$.

By Equation (4.16), a computation with Mathematica (cf. [Wol03]) shows that

$$\begin{aligned}
P_{p,q}(1,0) &= \frac{2p+3q}{5} + \frac{-p+q}{5}\zeta_5^2 + \frac{-p+q}{5}\zeta_5^3\\
P_{p,q}(1,1) &= \frac{p-q}{5} + \frac{3p+2q}{5}\zeta_5 + \frac{p-q}{5}\zeta_5^2\\
P_{p,q}(1,2) &= \frac{p-q}{5}\zeta_5 + \frac{3p+2q}{5}\zeta_5^2 + \frac{p-q}{5}\zeta_5^3\\
P_{p,q}(1,3) &= \frac{-p+q}{5} + \frac{-p+q}{5}\zeta_5 + \frac{2p+3q}{5}\zeta_5^3\\
P_{p,q}(1,4) &= -\frac{2p+3q}{5} - \frac{3p+2q}{5}\zeta_5 - \frac{3p+2q}{5}\zeta_5^2 - \frac{2p+3q}{5}\zeta_5^3\\
P_{p,q}(2,0) &= \frac{3p+2q}{5} + \frac{p-q}{5}\zeta_5^2 + \frac{p-q}{5}\zeta_5^3\\
P_{p,q}(2,1) &= \frac{-p+q}{5} + \frac{2p+3q}{5}\zeta_5 + \frac{-p+q}{5}\zeta_5^2\\
P_{p,q}(2,2) &= \frac{-p+q}{5}\zeta_5 + \frac{2p+3q}{5}\zeta_5^2 + \frac{-p+q}{5}\zeta_5^3\\
P_{p,q}(2,3) &= \frac{p-q}{5} + \frac{p-q}{5}\zeta_5 + \frac{3p+2q}{5}\zeta_5^3\\
P_{p,q}(2,4) &= -\frac{3p+2q}{5} - \frac{2p+3q}{5}\zeta_5 - \frac{2p+3q}{5}\zeta_5^2 - \frac{3p+2q}{5}\zeta_5^3\\
P_{p,q}(3,0) &= \frac{3p+2q}{5} + \frac{p-q}{5}\zeta_5^2 + \frac{p-q}{5}\zeta_5^3\\
P_{p,q}(3,1) &= \frac{-p+q}{5} + \frac{2p+3q}{5}\zeta_5 + \frac{-p+q}{5}\zeta_5^2\\
P_{p,q}(3,2) &= \frac{-p+q}{5}\zeta_5 + \frac{2p+3q}{5}\zeta_5^2 + \frac{-p+q}{5}\zeta_5^3\\
P_{p,q}(3,3) &= \frac{p-q}{5} + \frac{p-q}{5}\zeta_5 + \frac{3p+2q}{5}\zeta_5^3\\
P_{p,q}(3,4) &= -\frac{3p+2q}{5} - \frac{2p+3q}{5}\zeta_5 - \frac{2p+3q}{5}\zeta_5^2 - \frac{3p+2q}{5}\zeta_5^3\\
P_{p,q}(4,0) &= \frac{2p+3q}{5} + \frac{-p+q}{5}\zeta_5^2 + \frac{-p+q}{5}\zeta_5^3\\
P_{p,q}(4,1) &= \frac{p-q}{5} + \frac{3p+2q}{5}\zeta_5 + \frac{p-q}{5}\zeta_5^2\\
P_{p,q}(4,2) &= \frac{p-q}{5}\zeta_5 + \frac{3p+2q}{5}\zeta_5^2 + \frac{p-q}{5}\zeta_5^3\\
P_{p,q}(4,3) &= \frac{-p+q}{5} + \frac{-p+q}{5}\zeta_5 + \frac{2p+3q}{5}\zeta_5^3\\
P_{p,q}(4,4) &= -\frac{2p+3q}{5} - \frac{3p+2q}{5}\zeta_5 - \frac{3p+2q}{5}\zeta_5^2 - \frac{2p+3q}{5}\zeta_5^3.
\end{aligned}$$

If $\Sigma(K_{r,s})$ is equal to $\Sigma(K_{p,q})$, then there exist integers a and b such that

$$\tau^{\varphi_1}(\Sigma(K_{r,s}), \mathfrak{t}_{r,s}) = P_{p,q}(a, b).$$

Since the only $P_{p,q}(a, b)$ such that the non-zero coefficients are the ones of 1, ζ_5^2 and ζ_5^3 are $P_{p,q}(1, 0)$, $P_{p,q}(2, 0)$, $P_{p,q}(3, 0)$ and $P_{p,q}(4, 0)$, $\tau^{\varphi_1}(\Sigma(K_{r,s}), \mathfrak{t}_{r,s})$ must be equal to one of them. Moreover, since $P_{p,q}(1, 0) = P_{p,q}(4, 0)$ and $P_{p,q}(2, 0) = P_{p,q}(3, 0)$, $\tau^{\varphi_1}(\Sigma(K_{r,s}), \mathfrak{t}_{r,s})$ must be equal to either $P_{p,q}(1, 0)$ or $P_{p,q}(2, 0)$. The former case leads to $(p, q) = (r, s)$, whereas the latter leads to $(p, q) = (s, r)$. □

Remark. Roughly speaking, Lemma 4.17 does not allow one to distinguish $K_{p,q}$ from $K_{q,p}$. Indeed, it can happen that $K_{p,q} = K_{q,p}$. For example, $K_{0,1} = K_{1,0}$, $K_{0,-1} = K_{-1,0}$, $K_{1,-1} = K_{-1,1}$, $K_{0,2} = K_{2,0}$ and $K_{0,-2} = K_{-2,0}$. These equalities are all proved in Appendix A (see Figure A.1).

4.4.2 The main result

In this section we will prove the main result of the present work, which is stated in the following theorem.

Theorem 4.18. *Suppose that, for some integers p_0 and q_0, Kanenobu's knot K_{p_0,q_0} is thin. Then there exists some $M \in \mathbb{Z}$ such that the families*

$$\widetilde{\mathcal{F}}_M^+(p_0, q_0) = \{K_{p_0+2n,q_0-2n} \,|\, n > M,\, n \not\equiv p_0 - q_0 \pmod 5\}$$
$$\widetilde{\mathcal{F}}_M^-(p_0, q_0) = \{K_{p_0+2n,q_0-2n} \,|\, n < -M,\, n \not\equiv p_0 - q_0 \pmod 5\}$$

are infinite families of non-quasi-alternating thin knots.

Moreover, all knots in $\widetilde{\mathcal{F}}_M^+$ (resp. $\widetilde{\mathcal{F}}_M^-$) have the same Khovanov, odd-Khovanov and knot Floer homologies.

Proof. By Theorem 4.3 all the knots in $\mathcal{F} = \{K_{p_0+2n,q_0-2n}\}_{n\in\mathbb{Z}}$ have identical homological invariants, therefore so do the knots in the subsets $\widetilde{\mathcal{F}}_M^+$ and $\widetilde{\mathcal{F}}_M^-$. Thus, the last statement of the theorem is clear.

Now consider the four subfamilies $\mathcal{F}_{[j]}$, for $j \not\equiv p_0 - q_0 \pmod 5$. The knots in each subfamilies are $K_{p_0+10n+2j,q_0-10n-2j}$. Recall that by Corollary 4.4 all the knots in $\mathcal{F}$ are thin.

Focus on a particular j. The φ_0-torsion of the branched double cover of each knot in $\mathcal{F}_{[j]}$ vanishes (cf. Section 4.3.1). By Equation (4.16), the φ_1-torsion varies linearly in n in a non-trivial way. Finally, by Corollary 4.16, the φ_2-torsion is the same for all knots in $\mathcal{F}_{[j]}$. Hence, by Equation (3.20), the maximal abelian torsion

$$\tau(\Sigma(K_{p_0+10n+2j,q_0-10n-2j}), \mathfrak{t}_{p_0+10n+2j,q_0-10n-2j}) \in \mathbb{Q}\,[H]$$

varies linearly in n, and therefore so do all its rational coefficients. Since the sum of all coefficients vanishes (because it is the φ_0-torsion), there exist elements h_+ and h_- in H such that

$$\tau(\Sigma(K_{p_0+10n+2j,q_0-10n-2j}), \mathfrak{t}_{p_0+10n+2j,q_0-10n-2j}, h_+) \to -\infty \quad \text{if } n \to +\infty$$
$$\tau(\Sigma(K_{p_0+10n+2j,q_0-10n-2j}), \mathfrak{t}_{p_0+10n+2j,q_0-10n-2j}, h_-) \to -\infty \quad \text{if } n \to -\infty$$

Up to changing the $\mathrm{Spin}^{\mathbb{C}}$ structure, this implies, by Corollary 4.8, that for $n \to +\infty$ (resp. $n \to -\infty$), the correction term

$$d(\Sigma(K_{p_0+10n+2j,q_0-10n-2j}), \mathfrak{t}_{p_0+10n+2j,q_0-10n-2j})$$

tends to $-\infty$. Hence, by Theorem 2.10, for $n \gg 0$ (resp. for $n \ll 0$) $K_{p_0+10n+2j,q_0-10n-2j}$ is not quasi-alternating.

Since the previous fact holds for each subfamily $\mathcal{F}_{[j]}$ (with $j \not\equiv p_0 - q_0 \pmod 5$), up to choosing n big enough (resp. small enough), it holds also for the whole family $\widetilde{\mathcal{F}}$. $\square$

4.5 The starting step

Theorem 4.18 does not provide itself families of non-$\mathcal{QA}$ thin knots, but a thin knot is required to 'trigger' the theorem (recall the crucial hypothesis that K_{p_0,q_0} is thin).

In [GW11], Greene and Watson use as trigger the knot $K_{0,3} = 11^n_{50}$, which is the first non-$\mathcal{QA}$ thin knot to be discovered (cf. [Gre10]). Of course it works and produces the two families $\widetilde{\mathcal{F}}^+_M(0,3)$ and $\widetilde{\mathcal{F}}^-_M(0,3)$.

Remark. Note that a priori there is no reason for the knots in the families $\widetilde{\mathcal{F}}^+_M(p_0,q_0)$ and $\widetilde{\mathcal{F}}^-_M(p_0,q_0)$ to be different, because, as it was shown, the torsion does not distinguish between $K_{p,q}$ and $K_{q,p}$.

By Lemma 4.2, also the knot $K_{0,-3} = (11^n_{50})^r$ is thin, so it can be used as a trigger.

Other thin Kanenobu's knots are $K_{-2,0}$, $K_{-1,-1}$, $K_{-1,0}$, $K_{0,-1}$, $K_{-1,1}$, $K_{0,0}$, $K_{0,1}$, $K_{1,0}$, $K_{0,2}$ and $K_{1,1}$.

$K_{-2,0}$ is the knot 10^r_{137} (for the name of the knots we refer to the table at the end of [Rol76]), which is $\mathcal{QA}$ due to a result by Champanerkar and Kofman (cf. [CK09]), hence it is thin (cf. Theorem 2.3). The same argument holds for $K_{0,2}$, which is the knot 10_{137}.

$K_{-1,-1}$ is the knot 10_{155}, which was shown to be $\mathcal{QA}$ by Baldwin in [Bal08], hence it is thin. The same argument holds for $K_{1,1}$, which is the knot 10^r_{155}.

The knot $K_{0,0}$ is the connected sum $4_1\#4_1$, which admits an alternating diagram. Hence it is thin by [OS05, Lemma 3.2].

Finally, the knots $K_{-1,0} = 8_8$, $K_{0,-1} = 8_8$, $K_{-1,1} = 8_9$, $K_{0,1} = 8^r_8$ and $K_{1,0} = 8^r_8$ are alternating, so they are all thin.

Hence, each of the previous knots gives rise to the two families of non-$\mathcal{QA}$ thin knots $\widetilde{\mathcal{F}}^+_M(p_0,q_0)$ and $\widetilde{\mathcal{F}}^-_M(p_0,q_0)$.

Remark. All the isotopies between the above knots and their equivalent knots in the Rolfsen table are proved in Appendix A.

Remark. Before the work of Greene and Watson, the only known non-$\mathcal{QA}$ thin knot was $11^n_{50} = K_{0,3}$ (cf. [Gre10]). Then, starting from this knot, Greene and Watson in [GW11] constructed an infinite family of non-$\mathcal{QA}$ thin knots. The new families of non-$\mathcal{QA}$ thin knots found here were constructed without starting from already known non-$\mathcal{QA}$ thin knots. Therefore, the existence of each of these families provides an alternative argument for the existence of non-$\mathcal{QA}$ thin knots.

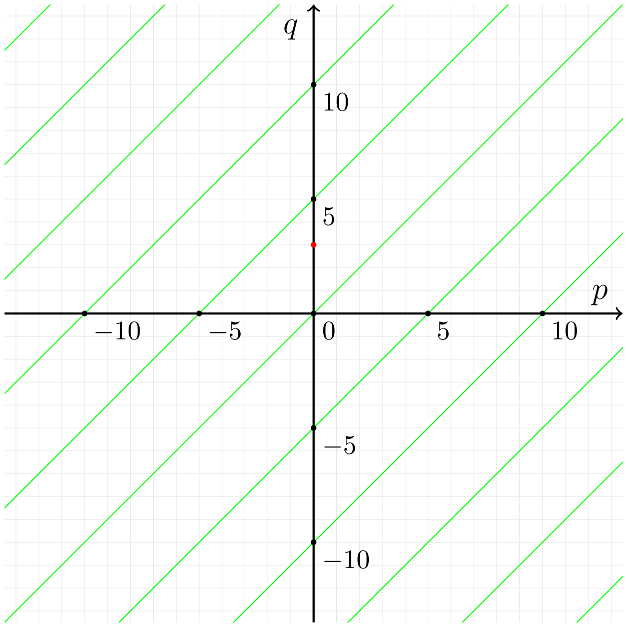


Figure 4.13: After [Gre10] the only known non-$\mathcal{QA}$ thin knot is the knot $K_{0,3} = 11^n_{50}$.

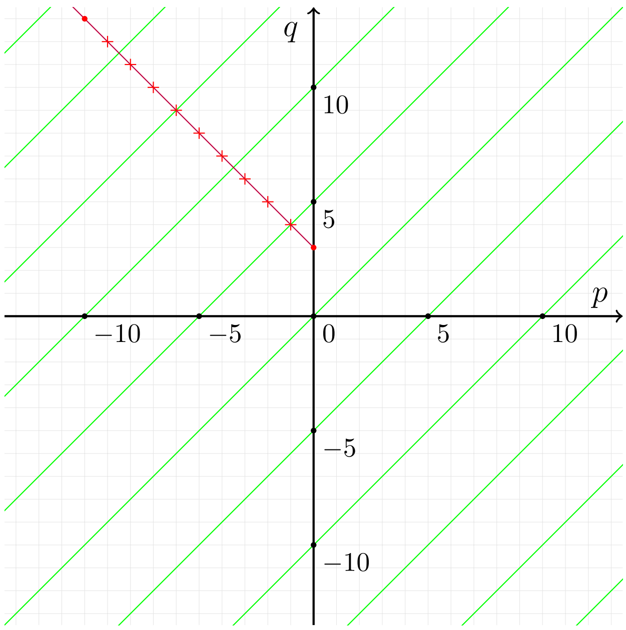


Figure 4.14: Greene and Watson in [GW11] proved that the family of knots obtained by taking on the purple line one knot every ten, starting from the knot $K_{0,3} = 11^n_{50}$ and going upwards and leftwards, contains infinite non-$\mathcal{QA}$ thin knots.

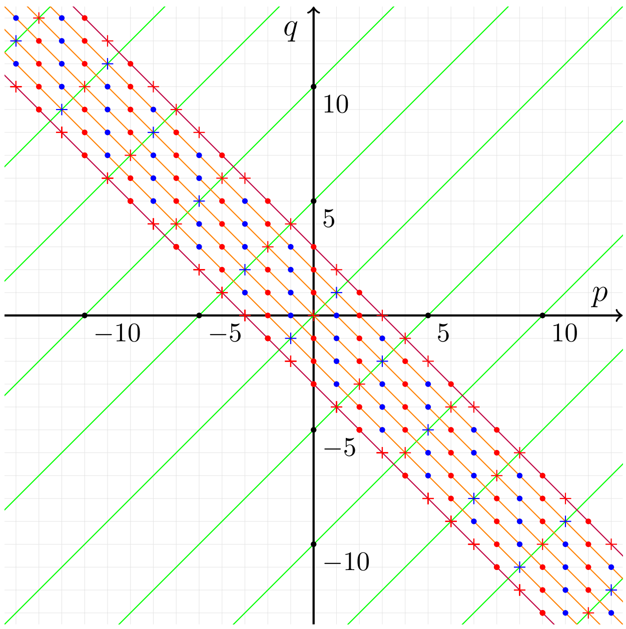


Figure 4.15: In this work we proved that both families lying on an orange line obtained by taking one knot every two (except the knots lying on the green lines) contain infinitely many non-$\mathcal{QA}$ thin knots in both directions. Moreover, the family lying on the purple line passing through $(0, 3)$ (resp. $(0, -3)$) obtained by taking one knot every two, starting from $K_{0,3}$ (resp. $K_{0,-3}$), excluding the knots lying on the green lines, contains infinitely many non-$\mathcal{QA}$ thin knots in both directions.

Appendix A

Tables of isotopies

In Section 4.5 some thin knots among Kanenobu's knots were detected in order to apply Theorem 4.18 to the family that they belong to. It was claimed that some knots $K_{p,q}$ were equivalent to knots of the Rolfsen table (cf. [Rol76]), but it was not proved.

In this appendix isotopies between Kanenobu's knots $K_{p,q}$ with small $|p|$ and $|q|$ and their equivalent knots in the Rolfsen table are shown.

Figure A.1 shows the equivalent knots in the Rolfsen table for small values of $|p|$ and $|q|$.

		11^n_{50}		
		10_{137}		
	8_9	8^r_8	10^r_{155}	
10^r_{137}	8_8	$4_1\#4_1$	8^r_8	10_{137}
	10_{155}	8_8	8_9	
		10^r_{137}		
		$(11^n_{50})^r$		

Figure A.1: The knots $K_{p,q}$ for small values of $|p|$ and $|q|$ in the notation of [Rol76]. On the horizontal axis we put the number p, whereas on the vertical one we put the number q. The central knot $4_1\#4_1$ corresponds to the knot $K_{0,0}$.

Each section of this appendix deals with one of the knots in Figure A.1 and provides an isotopy from the diagram as in Figure 4.1 to the diagram as in the Rolfsen table. By Lemma 4.2, for each p and q it is sufficient to provide an isotopy only for one of the knots $K_{p,q}$ and $K_{-p,-q}$.

We will not deal with the case of $K_{0,3} = 11^n_{50}$, for which we refer the reader to [Gre10].

A.1 The knot $K_{0,0}$

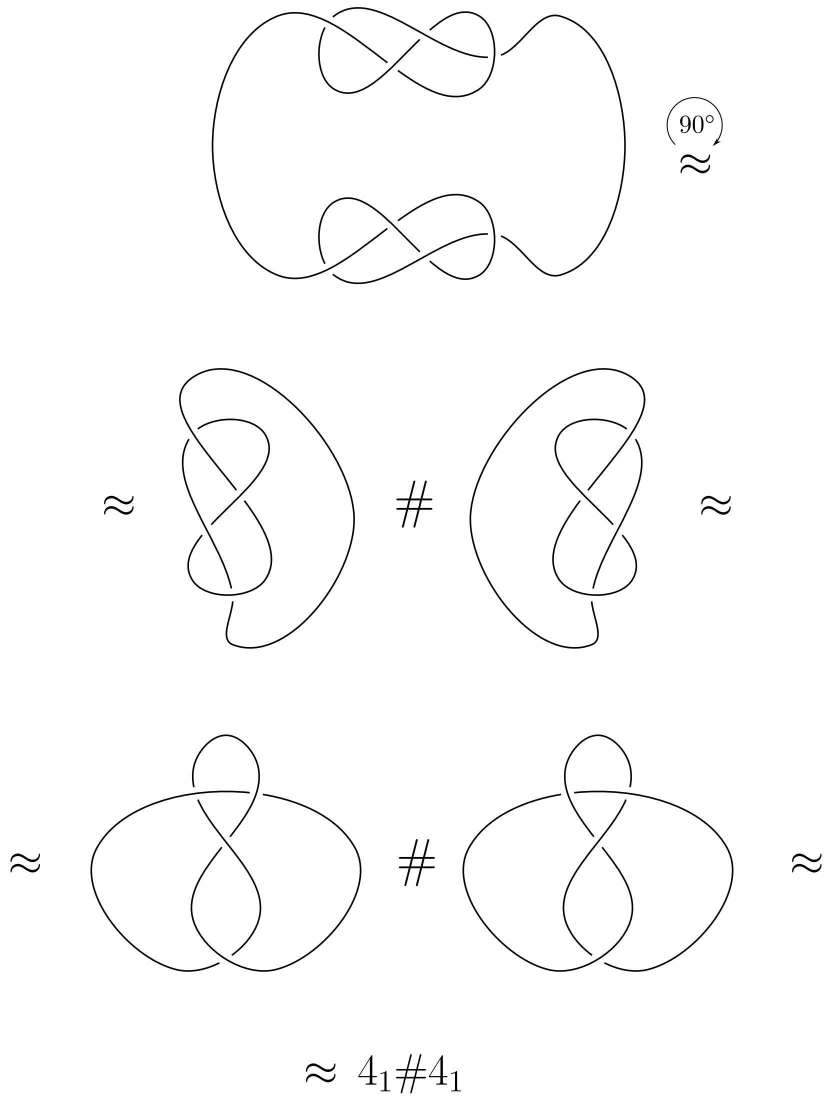


Figure A.2: Isotopy showing that $K_{0,0}$ is equivalent to the knot $4_1\#4_1$. Note that the figure-eight knot $\mathrm{F8} = 4_1$ is amphicheiral, i.e. $4_1 \approx 4_1^r$.

A.2 The knot $K_{1,0}$

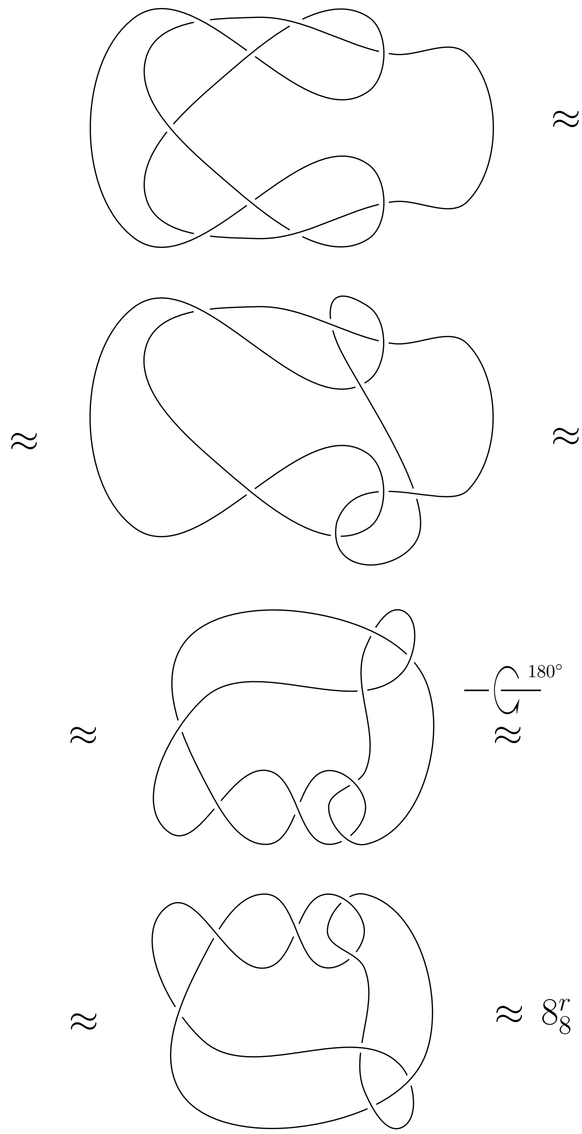


Figure A.3: Isotopy showing that $K_{1,0}$ is equivalent to the knot 8_8^r.

A.3 The knot $K_{0,1}$

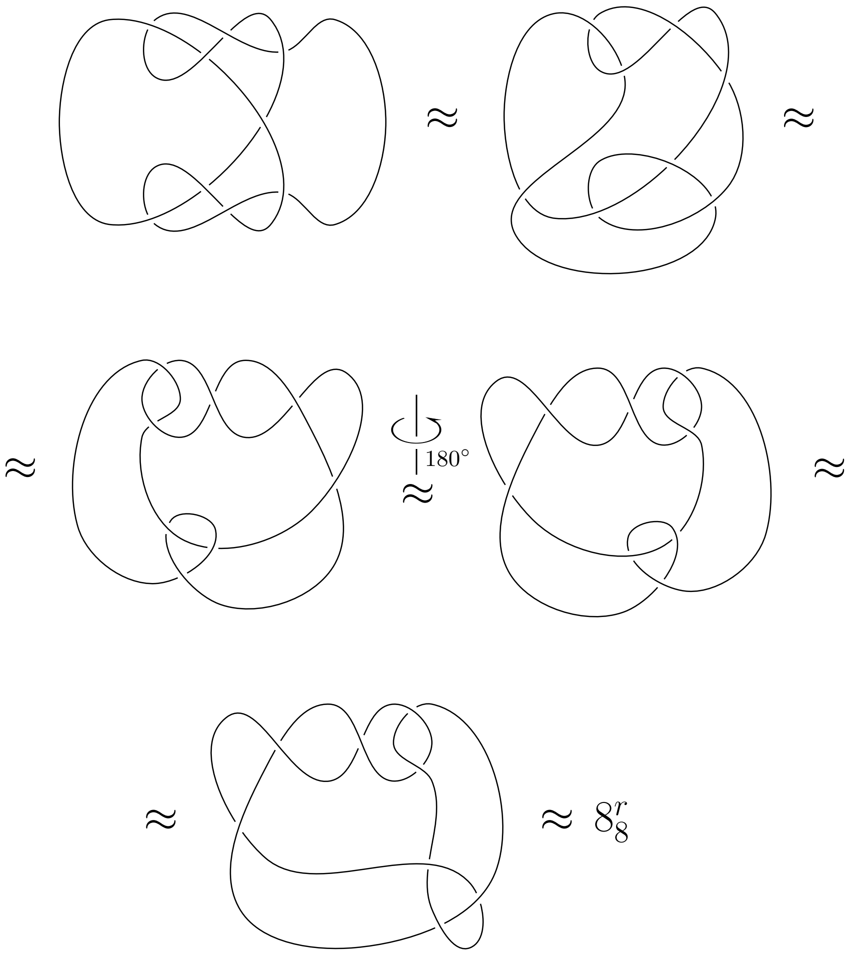


Figure A.4: Isotopy showing that $K_{0,1}$ is equivalent to the knot 8_8^r.

A.4 The knot $K_{1,-1}$

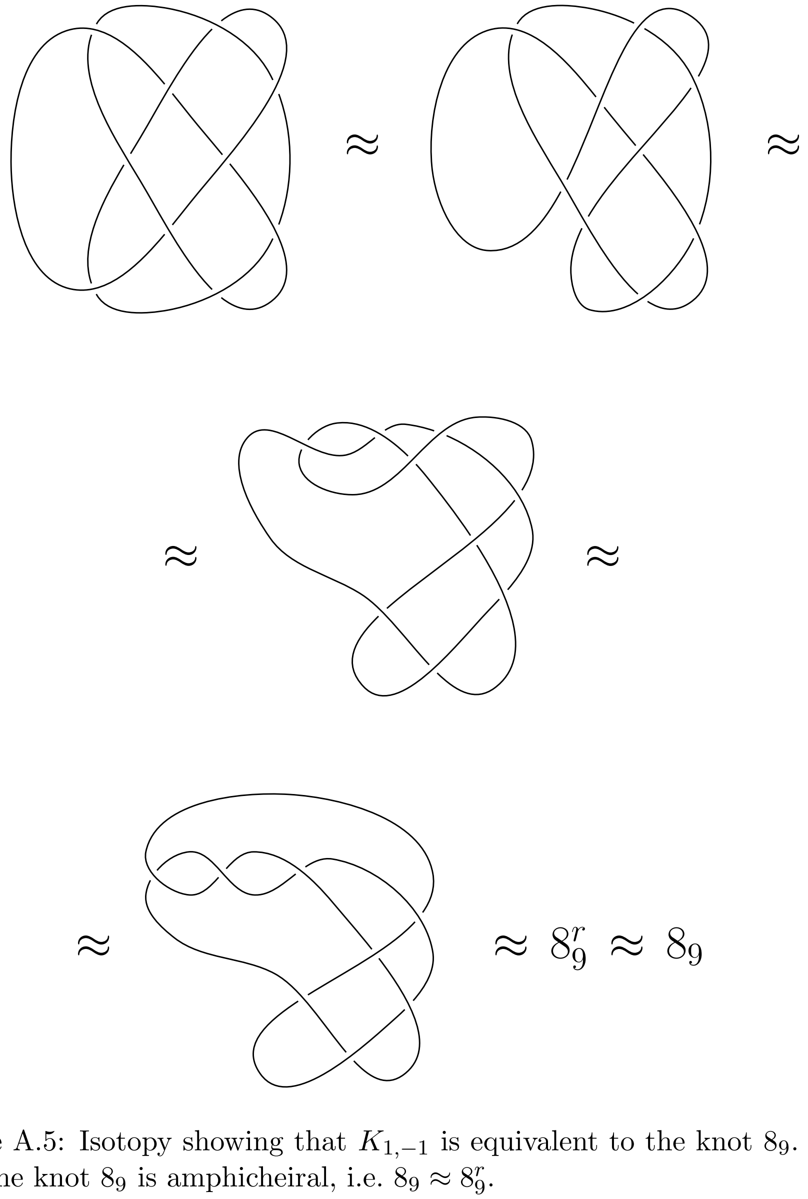


Figure A.5: Isotopy showing that $K_{1,-1}$ is equivalent to the knot 8_9. Note that the knot 8_9 is amphicheiral, i.e. $8_9 \approx 8_9^r$.

A.5 The knot $K_{2,0}$

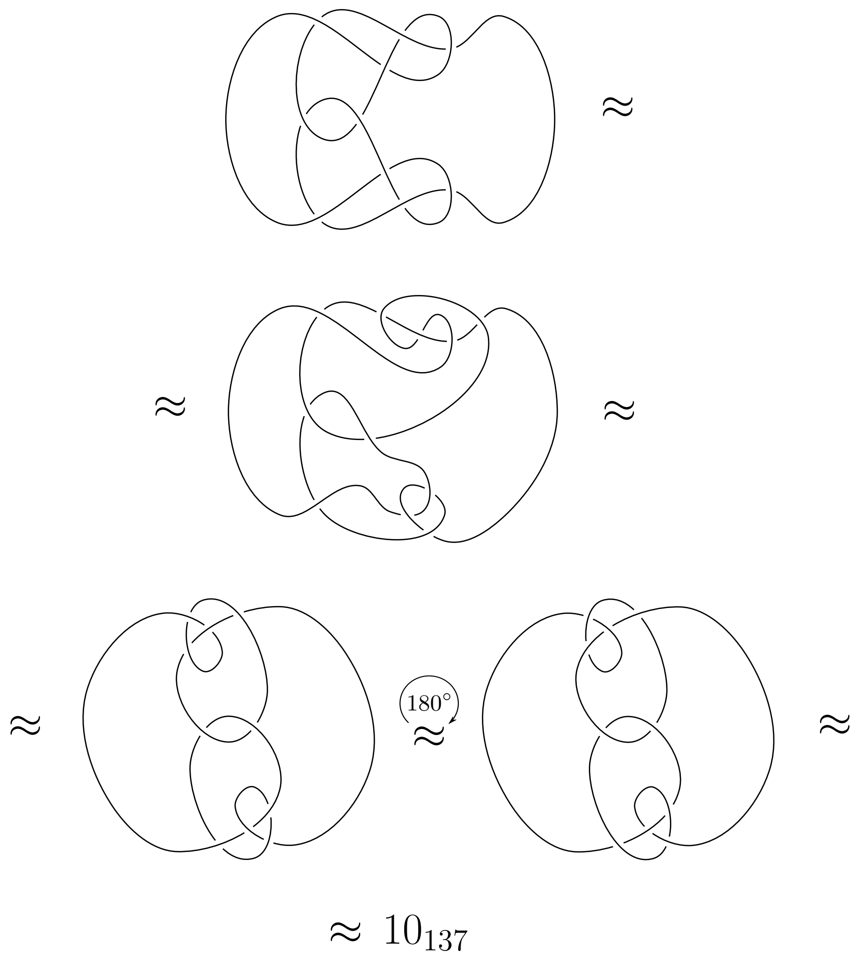


Figure A.6: Isotopy showing that $K_{2,0}$ is equivalent to the knot 10_{137}.

A.6 The knot $K_{1,1}$

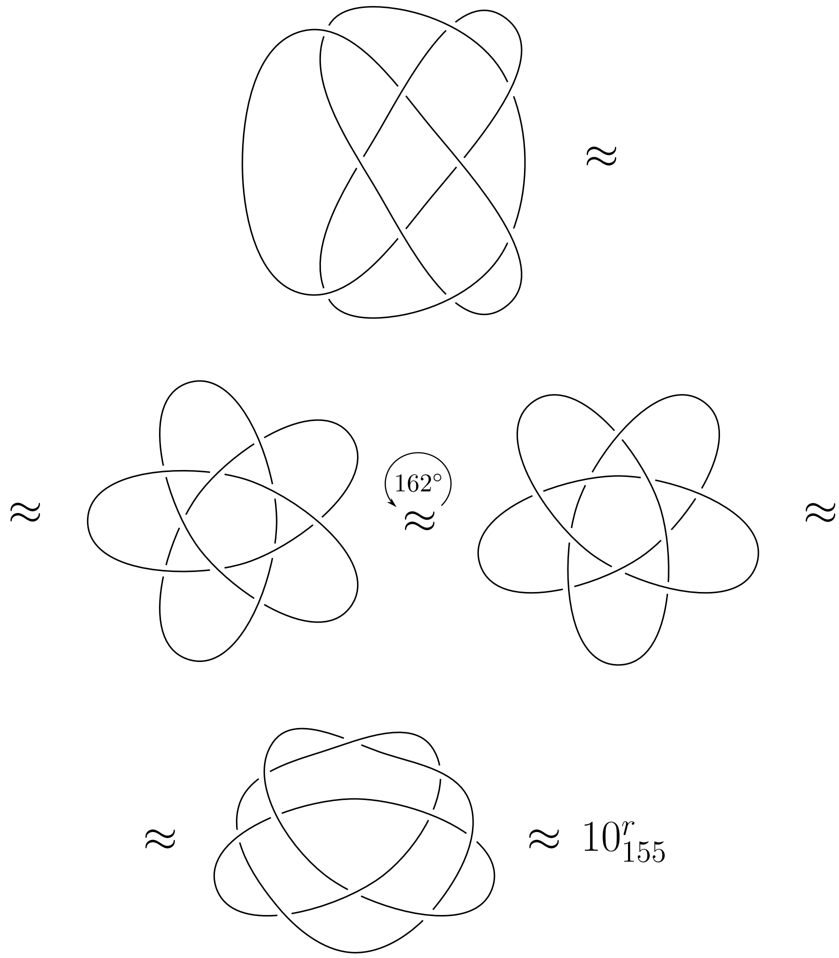
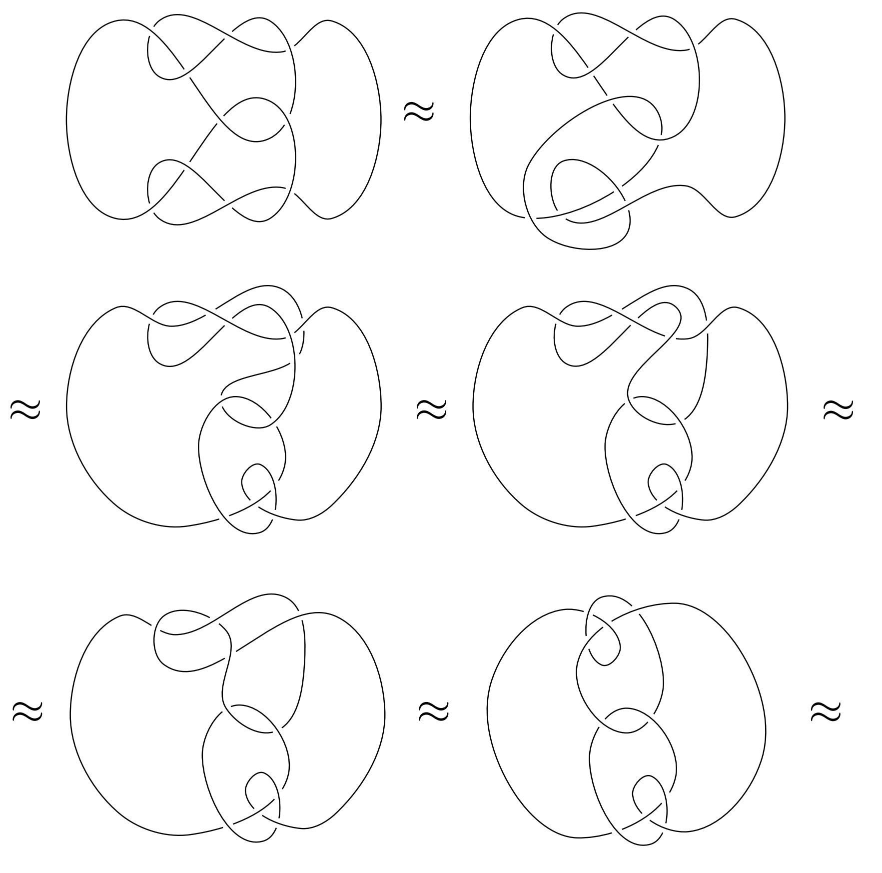


Figure A.7: Isotopy showing that $K_{1,1}$ is equivalent to the knot 10^r_{155}.

A.7 The knot $K_{0,2}$



$$\approx 10_{137}$$

Figure A.8: Isotopy showing that $K_{0,2}$ is equivalent to the knot 10_{137}.

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